Modelling Zombie Attack Using SIR Infectious Disease Models

MA3999

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Abstract: The aim of this report is to use SIR infectious disease modelling methods to model the spread of an outbreak of zombie infection. Conditions will be found which allow a human population to overcome the zombie threat, and prevent the outbreak escalating to that of a higher classification.

Project for MA3999

Title: Modelling zombie attack using SIR infectious disease models

Supervisor: D. Brown

Summary: SIR models have been used to model the spread of infectious diseases, such as AIDS and Ebola, and have more recently (with much publicity) been turned to zombie attack. The basic model is a susceptible-infected-removed three population model which is modified by a set of three coupled ordinary differential equations. However, more complicated variations involving extra populations, and interactions between populations tend to be used to better model real (or fantasy) world situations. In this project, the student would investigate SIR models, particularly the SZR application to zombie attack and its extensions (SIZR, etc). Further extensions to the model can be devised to investigate different human response strategies (such as fight back, hiding, and so on), and an ambitious student could extend this further by considering spatial terms in a coupled PDE model (which could be solved numerically using an explicit method).

Pre-requisites: MA2151 Mathematical Methods

Co-requisites: MA3001 Differential Equations

To pass (40%) the student must:

1. demonstrate understanding of three-population SIR models and their application to SZR models;
2. be able to perform a standard analysis (e.g., steady-state and stability) on this system;
3. be able to extract meaningful information about the physical system from the analysis;
4. submit a competent written report;

To obtain a mark between 41% and 69% the student must be able to do the above, plus:

1. demonstrate understanding of how the three-population model can be extended to higher population variants;
2. be able to perform standard analysis to these higher-population models;

3. be able to extract meaningful information about the physical system from these extended models;
4. explain the analysis and implications in a formal report written in good English;

To obtain a mark above 70% the student must be able to do all of the above, plus:

1. be able to develop new extensions to the model based on reasonable assumptions (this may be the inclusion of new population types, or the addition of spatially dependent terms);
2. analyse these extensions and extract detailed information about the physical system from the extended model.

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1 Introduction

A zombie is an animated human corpse that feeds on living human flesh. They are neither the work of magic or any other supernatural force. They are the result of a virus. Given that a zombie is a corpse, it has no life to end, thus killing one is impossible. However, this does not mean that they are invincible, they can be destroyed (Brooks, 2009). The undead have become a prominent figure in the horror genre of many forms of entertainment (Romero, 2005; Valve, 2008), and this report considers the potential outcomes were science fiction to become science fact. The aim will be to find a means of preventing the potential occurrence of a Class 4 outbreak, which would bring the end to human society and result in the total loss of all human life. Given that zombies are the result of a virus, this report will look at SIR infectious disease models, and apply them to an outbreak of zombie infection.

One report which models an outbreak of zombie infection in this manner, suggests that only quick and aggressive attacks can overcome the zombie threat (Smith? et al, 2009). However, a number of assumptions made within this work could be considered flawed, since one such assumption allows a single zombie to reanimate an infinite number of times. The models from the report will be presented and analysed to determine the stability of the steady states. Numerical simulations will be used to show the potential outcome of an outbreak under the modelling assumptions. Modifications will then be made to the models using alternative assumptions in regards to the population. Extensions to the models will be introduced, allowing the inclusion of a blaze class who's population is capable of destroying the zombie horde at an increased rate. It will be shown that, unless humanity is able to destroy the zombie horde at a sufficient rate, zombies will overwhelm the population with the potential to escalate into an outbreak of apocalyptic proportions.

2 Literature Review

As stated in section 1, this report will consider a zombie to be an animated human corpse that feeds on living human flesh. Hence, it will be assumed that a zombie would not attack other zombies, since neither of them possess living flesh, nor would a zombie attack a species other than a human. The zombie is often portrayed as a mindless being with no ability to feel emotion or pain, nor do they feel the effects of fatigue. However, the zombie is fundamentally a corpse lacking many of the bodily functions of a living human (Brooks, 2009). This is perhaps humanities greatest advantage over the living dead, since the dead possess no regenerative capabilities. Thus, if a zombie sustains damage the effects will be permanent. Whilst there are many ways to kill a human, the only way to successfully destroy a zombie is to destroy its brain. Hence, despite the seemingly supernatural capabilities of the zombie virus, the lack of a healing factor will mean that the brain cannot be repaired, resulting in the permanent destruction of the zombie.

The zombie virus is transferred through bodily fluids, the most common of which being saliva. Thus, when a zombie bites a susceptible individual, saliva covers the wound, allowing the virus immediate entry into the bloodstream of the unfortunate victim, infecting them instantaneously. The process through which an individual becomes a zombie will be referred to as zombification. There is no natural immunity to the zombie virus, it is one hundred percent communicable. Upon infection, there is on average a period of 24 hours before the infected individual becomes a zombie. During this time, many of the individual's bodily functions will shut down, until they fall into a coma, and later reanimate.

A crucial area of research with most infectious diseases is the concept of either a cure, or a vaccination against the disease. Given the aggressive nature of the virus, vaccination is impossible. Even the smallest dose would result in the individual becoming a zombie. The only reasonable way to cure an individual of infection would be through the use of a counter virus. However, this would have to be applied prior to the zombification of the infected individual, since there is no cure for death which is an unfortunate side effect of becoming a zombie.

Outbreaks of zombie infection fall into four classifications (Brooks, 2009).

A Class 1 outbreak is considered a low level outbreak. The number of zombies present throughout the duration of this class of outbreak would be expected to range between 1 and 50, with the number of human fatalities reaching as high as 100. The duration of such an outbreak would be expected to be in the range of 24 hours and 1 month.

A Class 2 outbreak is defined when the number of zombies ranges between 20 and 500. In such an outbreak, the number of human casualties will be higher than a Class 1 outbreak, with the number of fatalities as high as 800. The duration of a Class 2 outbreak may only be as long as that of a Class 1.

A Class 3 outbreak presents a large threat on a global scale. The zombie population may be as high as 5000, and the outbreak could span months if not years. This classification of outbreak would see an even larger number of human fatalities than either a Class 1 or a Class 2 outbreak.

The Class 4 outbreak is of apocalyptic proportions. The undead will outnumber human life, and society will fall. The human population of the world will plummet, and the zombie horde will be an ever present danger. It would prove difficult to sustain any form of life were such a scenario to occur. It is expected that an outbreak of this classification would result in the total loss of all human life.

This report will consider a population with the potential to escalate to a Class 2 outbreak. If the outbreak overwhelms the population, then it could result in a Class 4 outbreak of zombie infection. However, if the zombie horde can be eradicated prior to the outbreak obtaining the classification of a Class 2, then a Class 4 would not be possible and hence prevented. In order to model a scenario such

as this, some basic infectious disease models must first be considered, which a could then be applied to model an outbreak of zombie infection.

3 Infectious Disease Modelling

In order to model an epidemic of a zombie virus, one must first consider basic modelling methods for infectious diseases. In order to model an epidemic process, some assumptions must be made about the population which is affected, namely, the population dynamics. In an epidemic, the timescale of the outbreak of the disease is expected to be short in comparison to the lifetime of the susceptible host, hence it will often be assumed that natural birth and death rates can be neglected. For the purposes of modelling the epidemic, the population will be assumed to mix with one another homogeneously, without prejudice. This section will consider the behaviour of a virus, since this is the type of pathogen that the zombie infection is assumed to be caused by, as stated in section 2. Thus, any individual in the population either has the disease, or does not.

3.1 The SI Epidemic

A model for an SI epidemic is proposed as follows (Britton, 2005). The population of this epidemic consists of only a susceptible class, and an infective class. The susceptible class, S, contains the individuals in the population that are susceptible to contracting the virus. The infective class, I, contains the individuals of the population which have contracted the virus. Given that the disease is assumed to be a micro-parasitic virus, it will be assumed that the disease is contagious, and is spread by an interaction between a member of the infective class, and a member of the susceptible class. In this model, an infective may infect a susceptible with the disease, which moves the previously susceptible individual into the infective class, where the individual remains indefinitely. It will be assumed that the timescale of the epidemic is short compared to the hosts lifetime, allowing natural birth and death rates of the population to be neglected. This gives a closed population,

$$S(\tau) + I(\tau) = N, \tag{3.1.1}$$

where $S(\tau)$ and $I(\tau)$ are the numbers of susceptible and infective individuals at time τ respectively. Hence, the S and I classes satisfy the differential equations,

$$\frac{dS}{d\tau} = -f(S,I), \tag{3.1.2}$$

$$\frac{dI}{d\tau} = f(S,I), \tag{3.1.3}$$

where $f(S,I)$ is the rate at which new infections occur. A simple modelling assumption is that,

$$f(S,I) = (\beta N)\left(\frac{S}{N}\right)I, \tag{3.1.4}$$

where (βN) is the number of contacts sufficient for transmission of the infection made by an average member of the population per unit time, and $\left(\frac{S}{N}\right)$ is the probability that a random contact is made by an infective with a susceptible. Thus, the number of new infections per unit time, is given by,

$$f(S,I) = \beta SI. \tag{3.1.5}$$

Here, β is the pairwise infection rate, meaning the rate of infection per susceptible, per infective. This is the law of mass action (Britton, 2005). Thus, using equation 3.1.5, equations 3.1.2 and 3.1.3 become,

$$\frac{dS}{d\tau} = -\beta SI, \tag{3.1.6}$$

$$\frac{dI}{d\tau} = \beta SI, \tag{3.1.7}$$

respectively.

A disease such as this may be represented diagrammatically as shown in figure 1.

Figure 1: Flowchart representing the SI epidemic model.

Under these modelling assumptions, the steady states are found to be,

$$(S^*, I^*) = (0, N), \tag{3.1.8}$$

$$(S^*, I^*) = (N, 0), \tag{3.1.9}$$

where equation 3.1.9 is referred to as the disease free steady state, since the population of the infective class at this steady state is equal to zero. The steady state in equation 3.1.8 will be referred to as the endemic steady state, since the disease will remain prevalent in the population indefinitely. The Jacobian matrix may then be used to determine the stability of the steady states. The 2x2 Jacobian matrix is given as (Allen, 2007),

$$J = \begin{bmatrix} \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} \\ \frac{\partial m}{\partial S} & \frac{\partial m}{\partial I} \end{bmatrix}, \tag{3.1.10}$$

where $\frac{\partial g}{\partial S}$ is the partial derivative of some function g, with respect to the variable S. Hence define,

$$g(S,I) = \frac{dS}{d\tau} = -\beta SI, \tag{3.1.11}$$

$$m(S,I) = \frac{dI}{d\tau} = \beta SI, \tag{3.1.12}$$

so that the Jacobian matrix for the SI model is given by,

$$J = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S \end{bmatrix}. \tag{3.1.13}$$

Evaluating the Jacobian matrix in equation 3.1.13 at the disease free steady state gives,

$$J(N,0) = \begin{bmatrix} 0 & -\beta S \\ 0 & \beta S \end{bmatrix}, \tag{3.1.14}$$

which can then be used to determine the eigenvalues. The eigenvalues are found from calculating,

$$det(J(N,0) - \lambda I_2), \tag{3.1.15}$$

where I_n is the nxn identity matrix. Hence it is obtained that,

$$det(J(N,0) - \lambda I_2) = \begin{bmatrix} -\lambda & -\beta S \\ 0 & \beta S - \lambda \end{bmatrix} = 0,$$

$$det(J(N,0) - \lambda I_2) = -\lambda(\beta S - \lambda) = 0,$$

which gives that the eigenvalues are,

$$\lambda_1 = 0, \tag{3.1.16}$$

$$\lambda_2 = \beta S. \tag{3.1.17}$$

Hence the disease free steady state is unstable, since the eigenvalue λ_2, in equation 3.1.17 is positive, since $\beta > 0$ and $S > 0$, and for stability the eigenvalues must have negative real part (Britton, 2005). Evaluating the Jacobian matrix in equation 3.1.13 at the endemic steady state gives,

$$J(0,N) = \begin{bmatrix} -\beta N & 0 \\ \beta N & 0 \end{bmatrix},$$

which, by a similar derivation process as for the disease free steady state, has eigenvalues,

$$\lambda_1 = -\beta N, \tag{3.1.18}$$

$$\lambda_2 = 0, \tag{3.1.19}$$

and hence the endemic steady state is always stable, since λ_1, from equation 3.1.18, is always negative. Hence, this model determines that once a disease becomes present in the population, it will always spread and infect all individuals of the susceptible class. A simulation of an outbreak of infection given by this model is shown below in figure 2.

For the simulation in figure 2, it was assumed that a single infective individual was introduced into a population of 500 susceptibles. It was then assumed that a single infective individual in such a scenario would be able to make, on average, 2.5 contacts sufficient to transmit the infection per unit time. Hence, it is chosen that the parameter $\beta = 0.005$, such that $\beta SI = 2.5$, in a population with 500 susceptible individuals and just a single infective. In this model, time, τ, is measured in days. Hence figure 2 shows that the population would become entirely infected after approximately 9 days.

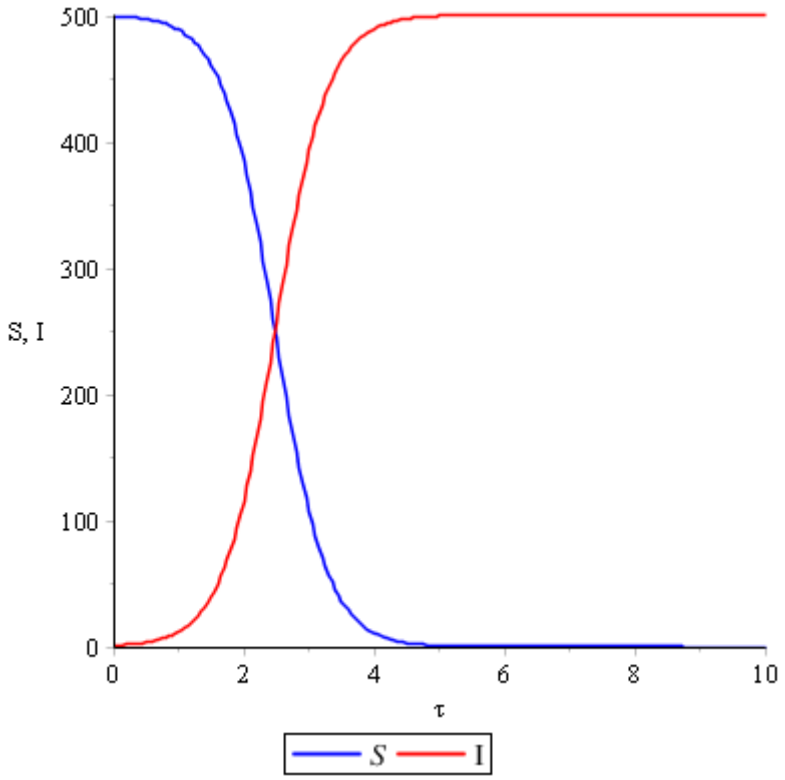

Figure 2: Simulation of the SI model showing the entire population becoming infected.

Appendix 1 shows the worksheet for Maple 17 which was used to calculate the steady states, Jacobian matrix, and perform the simulation for this model. All further models follow this example by a similar manner.

3.2 The SIS Epidemic

The previous SI disease model assumes that once an individual becomes infected, they remain so indefinitely. A basic model for an SIS infectious disease proposes that an infected individual may recover from the disease (Britton, 2005). However, assume that the infection conveys no immunity and hence the individual becomes susceptible to the disease on recovery, thus returning to the susceptible class. It will again be assumed that the modelling time scale is short in comparison to the life time of the host, hence natural birth and death rates may be neglected. This gives a closed population, as with the previous model, such that equation 3.1.1 is retained. However, S and I now satisfy the differential equations,

$$\frac{dS}{d\tau} = -f(S,I) + g(I), \tag{3.2.1}$$

$$\frac{dI}{d\tau} = f(S,I) - g(I), \tag{3.2.2}$$

where $f(S,I)$ is the rate at which new infections occur, and $g(I)$ is a term to allow an individual to recover from the infection. A simple model for $g(I)$ is,

$$g(I) = \mu I, \tag{3.2.3}$$

where the parameter μ is the rate of recovery from the infection. Hence, $\frac{1}{\mu}$ gives the average time taken for an individual to recover from the infection. Thus, using the same function for $f(S, I)$ as given in equation 3.1.5, and using equation 3.2.3, the equations 3.2.1 and 3.2.2 become,

$$\frac{dS}{d\tau} = -\beta SI + \mu I, \tag{3.2.4}$$

$$\frac{dI}{d\tau} = \beta SI - \mu I, \tag{3.2.5}$$

respectively. This disease may now be diagrammatically represented by figure 3 given below.

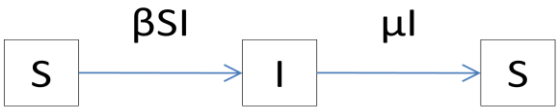

Figure 3: Flowchart for the SIS epidemic model.

The steady states of this model may be determined from equations 3.2.4 and 3.2.5 and are given as,

$$(S^*, I^*) = (N, 0), \tag{3.2.6}$$

$$(S^*, I^*) = \left(\frac{\mu}{\beta}, \frac{\beta N - \mu}{\beta}\right), \tag{3.2.7}$$

where equation 3.2.6 gives the disease free steady state, and equation 3.2.7 will be referred to as the endemic steady state for this model.

The Jacobian matrix for the SIS model is given by,

$$J = \begin{bmatrix} -\beta I & -\beta S + \mu \\ \beta I & \beta S - \mu \end{bmatrix}, \tag{3.2.8}$$

which, when evaluated at the disease free steady state, becomes,

$$J(N, 0) = \begin{bmatrix} 0 & -\beta N + \mu \\ 0 & \beta N - \mu \end{bmatrix}.$$

Next, calculating $\det(J(N, 0) - \lambda I_2)$ gives eigenvalues,

$$\lambda_1 = 0, \tag{3.2.9}$$

$$\lambda_2 = \beta N - \mu. \tag{3.2.10}$$

Thus, for the disease free steady state to be stable, both equations 3.2.9 and 3.2.10 must have a negative real part. Hence it is returned from equation 3.2.10 that, for stability,

$$\mu > \beta N, \tag{3.2.11}$$

meaning that the rate of recovery from the disease must be greater than the number of infectious contacts sufficient to transmit the infection made per unit time by an average member of the population, in order for the disease free steady state to be stable, and the infection to be eradicated.

Now evaluating the Jacobian matrix in equation 3.2.8 at the endemic steady state gives,

$$J\left(\frac{\mu}{\beta}, \frac{\beta N-\mu}{\beta}\right)=\begin{bmatrix}\mu-\beta N & 0\\ \beta N-\mu & 0\end{bmatrix}.$$

Next, calculating, $\det\left(J\left(\frac{\mu}{\beta}, \frac{\beta N-\mu}{\beta}\right)-\lambda I_2\right)$ gives the eigenvalues,

$$\lambda_1=\mu-\beta N, \tag{3.2.12}$$

$$\lambda_2=0. \tag{3.2.13}$$

Hence, in order for the endemic steady state to be stable, equation 3.2.12 gives that $\lambda_1<0$, thus the inequality,

$$\mu<\beta N, \tag{3.2.14}$$

must be satisfied. Thus meaning that if the rate of recovery is not sufficiently fast enough, then the disease will remain endemic in the population.

The results from equations 3.2.11 and 3.2.14 are supported by simulations of the model.

Figure 4 shows a simulation where equation 3.2.14 is satisfied, resulting in the disease remaining endemic in the population. As with the previous simulation, it was assumed that a single infective was introduced to a population of 500 susceptibles. Also, parameter $\beta=0.005$, and it was assumed that it would take in infective individual, on average, 0.6 days to recover from the disease. Hence, $\mu=\frac{5}{3}$. Thus, figure 4 shows that the endemic steady state is reached after approximately 15 days.

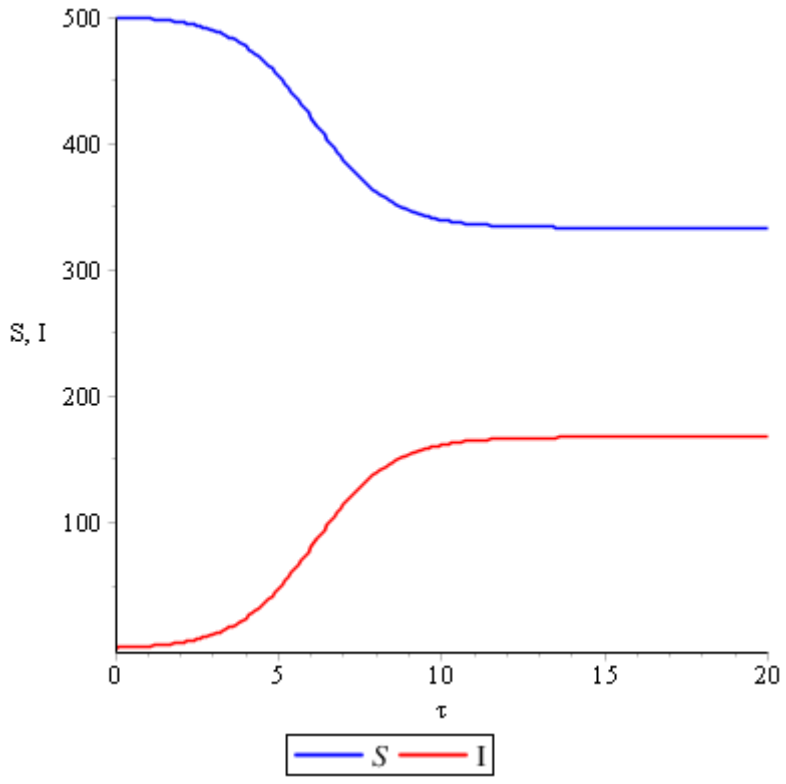

Figure 4: Simulation of the SIS model where the endemic steady state is reached.

Figure 5 shows a simulation of the SIS model where the disease free steady state is stable. This was achieved by assuming that an infective individual recovers from the disease after just 0.2 days on average. Hence it was assumed that $\mu = 5$, thus satisfying equation 3.2.11 since $\beta N = 2.5$. This simulation found that, under the stated assumptions, the disease is eradicated after approximately 4 days.

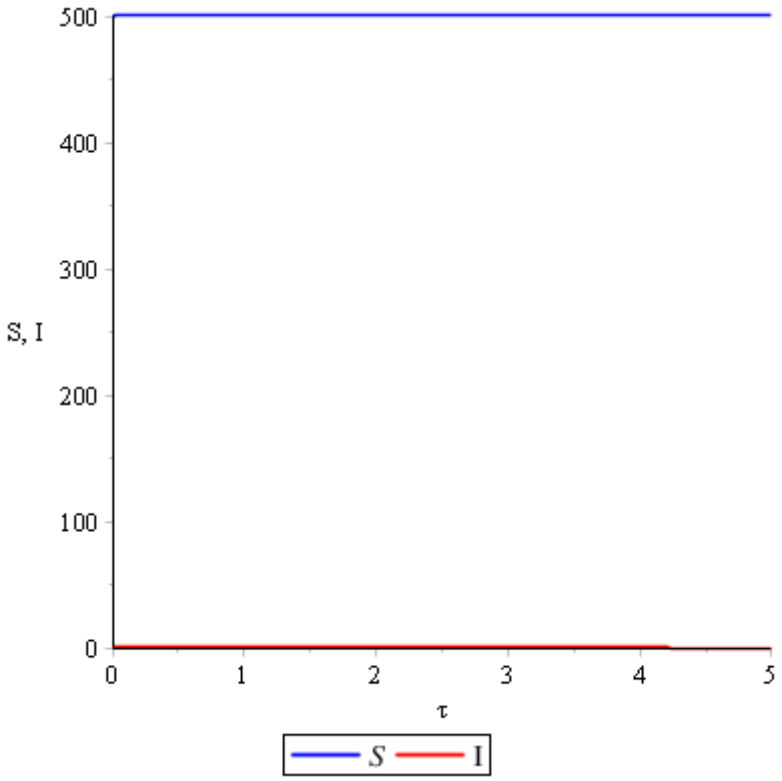

Figure 5: Simulation of the SIS model were the disease free steady state is stable.

3.3 The SIR Epidemic

A basic model for an SIR infectious disease will be introduced in this section (Britton, 2005). This model now assumes that upon leaving the infective class, an individual would have no further role to play in the transmission of the disease. This could be due to some immunity conveyed by the infection, death, or otherwise. These individuals will instead enter a removed class, R, and remain there indefinitely. Chicken Pox and Measles behave in this manner, as once the individual has recovered from the disease, they may not contract it again nor pass it on to a susceptible individual. As in section 3.1 and 3.2, the time scale of the outbreak will be assumed to be short, thus natural birth and death rates of the population will be neglected. This gives a closed population as with the previous models. However, equation 3.1.1 must now be modified due to the introduction of the R class into the model and becomes,

$$S(\tau)+I(\tau)+R(\tau)=N. \qquad (3.3.1)$$

The differential equations that S,I and R satisfy in this model are,

$$\frac{dS}{d\tau} = -\beta SI, \tag{3.3.2}$$

$$\frac{dI}{d\tau} = \beta SI - \mu I, \tag{3.3.3}$$

$$\frac{dR}{d\tau} = \mu I. \tag{3.3.4}$$

Hence the model can be represented as shown in Figure 6.

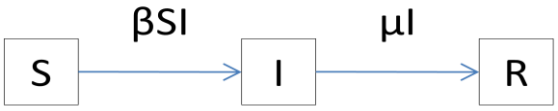

Figure 6: Flowchart for the SIR epidemic model.

In order to determine the steady states and stability of the SIR model, the differential equations in equations 3.3.2, 3.3.3 and 3.3.4 will now be renormalised, using the substitutions, $u = \frac{S}{N}, v = \frac{I}{N}$, $w = \frac{R}{N}, t = \mu\tau, R_0 = \frac{\beta N}{\mu}$, and $u + v + w = 1$. Thus u gives the percentage of the population which is susceptible to infection, v gives the infective percentage of the population, and w is the percentage of the population which lies in the removed class. The renormalised equations of this system are given by,

$$\frac{du}{dt} = -R_0 uv, \tag{3.3.5}$$

$$\frac{dv}{dt} = (R_0 u - 1)v, \tag{3.3.6}$$

$$\frac{dw}{dt} = v. \tag{3.3.7}$$

Given that w is not present in equations 3.3.5 and 3.3.6, these equations may be considered on their own. However, when considering the flow of these equations on the u, v plane, the u-axis is a nullcline, and hence any point on it is a steady state. Thus giving the steady state,

$$(u^*, v^*) = (u^*, 0). \tag{3.3.8}$$

The Jacobian matrix of equations 3.3.5 and 3.3.6 is given by,

$$J = \begin{bmatrix} -R_0 v & -R_0 u \\ R_0 v & R_0 u - 1 \end{bmatrix}, \tag{3.3.9}$$

which, when evaluated at the steady state in equation 3.3.8, returns the eigenvalues,

$$\lambda_1 = 0, \tag{3.3.10}$$

$$\lambda_2 = R_0 u - 1. \tag{3.3.11}$$

Thus, for stability of the disease-free steady state, $\lambda_2 < 0$, hence, $u < \frac{1}{R_0}$. Likewise, if $u > \frac{1}{R_0}$, then the disease-free steady state will be unstable. The phase planes for the different values of the parameter R_0 are given in figures 7 and 8. In figure 7 it was given that $R_0 = 1.5$, whilst in figure 8, $R_0 = 0$.

Figure 7 and 8 support the analysis that the disease will always be eradicated from the population in the SIR model, with some fraction of the population remaining susceptible after the outbreak. The percentage of the population which lies in the removed class at the steady state may be obtained from, $w = 1 - u^*$.

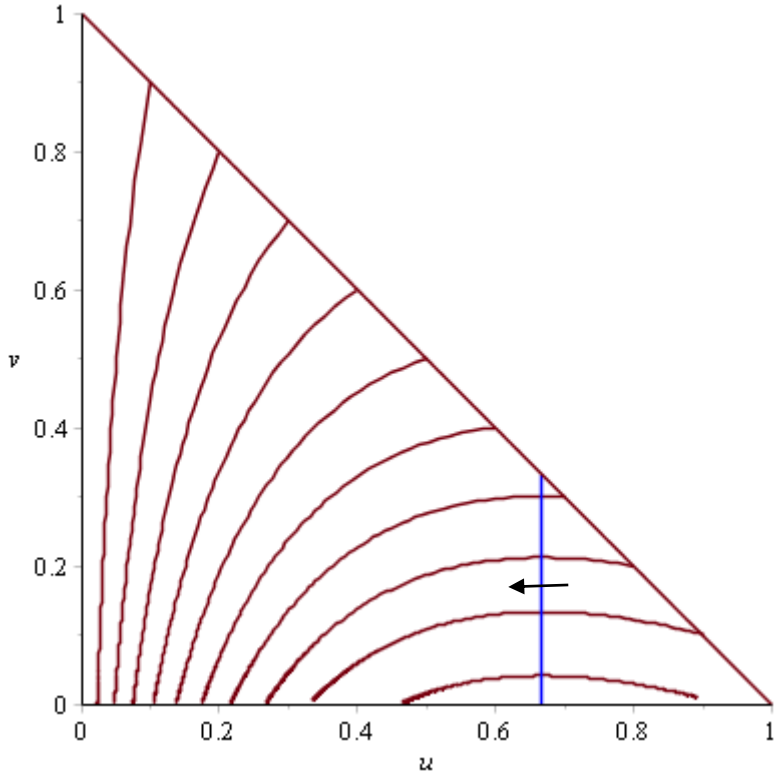

Figure 7: Phase plane for the SIR epidemic with R_0>1.

The differential equations in equations 3.3.2, 3.3.3, and 3.3.4 may also be used to simulate the SIR model. Assuming that a single infective is introduced into a population of 500 susceptibles gives the initial conditions, $S(0) = 500, I(0) = 1$, and $R(0) = 0$. The values chosen for parameters μ and β are retained from the simulation in figure 4. Hence, this simulation of the SIR model finds that the disease would be eradicated in approximately 30 days, with the majority of the population belonging to the R class at the steady state. This simulation is shown in figure 9.

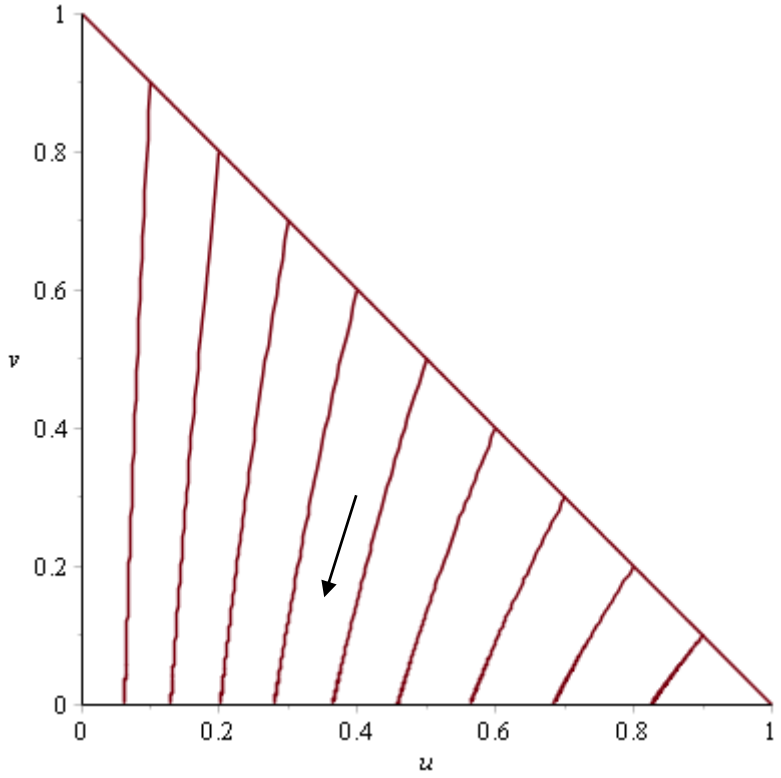

Figure 8: Phase plane plot for the SIR epidemic with R_0<1.

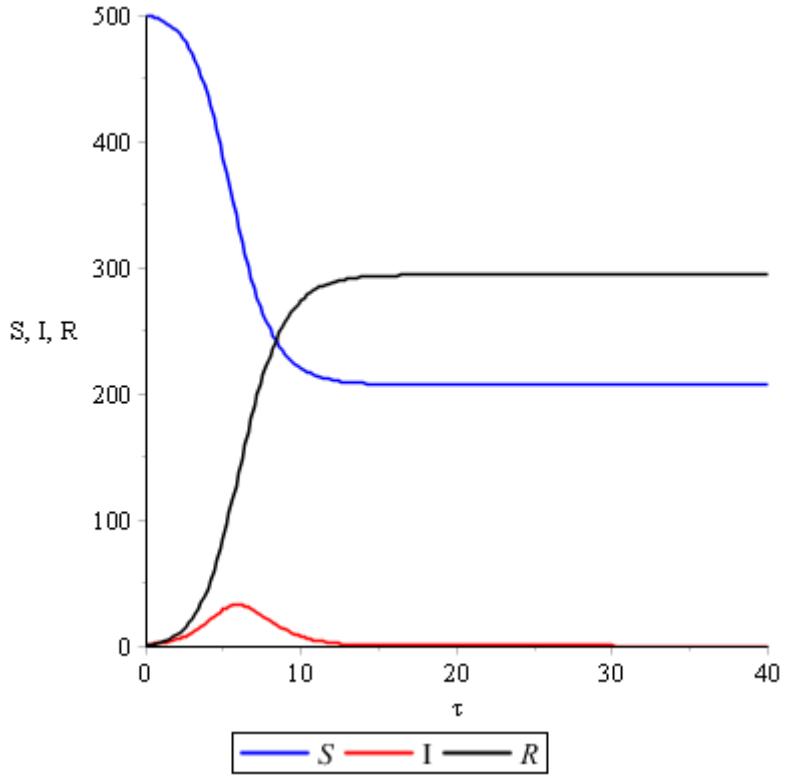

Figure 9: Simulation of an SIR epidemic where the disease free steady state is stable.

3.4 The SIR Endemic

Now a model for an SIR endemic is proposed (Britton, 2005). This model will consider an outbreak of a disease over a larger timescale, hence it is no longer reasonable to neglect the natural birth and death rates of the host. This will be called the endemic SIR model. However, since birth rates now being modelled, the birth rate is given by, $B = bN$, where N is given in equation 3.3.1. However, it is unreasonable to have both immune and deceased individuals entering the R class, as the model would then allow for the dead to continue to reproduce. Hence the dead will simply leave the model, leaving R as a class consisting only of immune individuals.

The inclusion of birth and death rates means that the population is no longer closed. However, the death rate of the population will be assumed to be equal to the birth rate such that, $b = d$, and hence the population size, N, will remain constant. This is shown by the differential equation,

$$\frac{dN}{d\tau} = \frac{dS}{d\tau} + \frac{dI}{d\tau} + \frac{dR}{d\tau} = 0. \tag{3.4.1}$$

This model will also assume that there is no disease related death, and that all births will enter the susceptible class, meaning that there is no vertical immunity conveyed by the population of the R class. Hence, the differential equations for S, I and R in this model are,

$$\frac{dS}{d\tau} = bN - \beta SZ - bS, \tag{3.4.2}$$

$$\frac{dI}{d\tau} = \beta SI - \mu I - bI, \tag{3.4.3}$$

$$\frac{dR}{d\tau} = \mu I - bR. \tag{3.4.4}$$

Thus the model may be represented as shown in Figure 10.

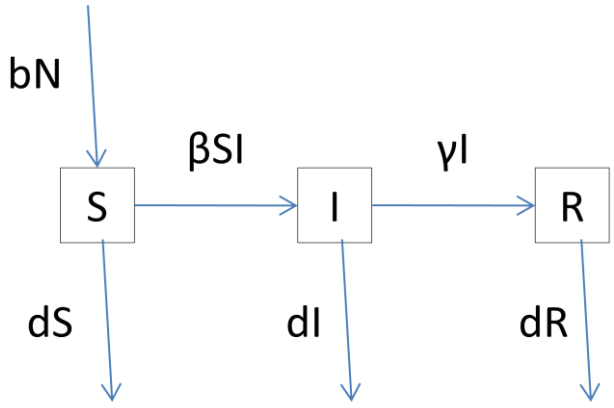

Figure 10: Flowchart of the SIR Endemic model with no disease related death.

This model can then be renormalised in order to analyse it, using the substitutions, $u = \frac{S}{N}, v = \frac{I}{N}$,

$w = \frac{R}{N}, t = \mu\tau, R_0 = \frac{\beta N}{\mu}$, and $u + v + w = 1$. Thus equations 3.4.2, 3.4.3, and 3.4.4 become,

$$\frac{du}{dt} = \frac{b(1-u)}{b+\mu} - R_0 uv, \tag{3.4.5}$$

$$\frac{dv}{dt} = (R_0 u - 1)v, \tag{3.4.6}$$

$$\frac{dw}{dt} = \frac{\mu v - bw}{b+\mu}, \tag{3.4.7}$$

and hence the steady states of these equations are,

$$(u^*, v^*, w^*) = (1,0,0), \tag{3.4.8}$$

$$(u^*, v^*, w^*) = \left(\frac{1}{R_0}, \frac{b(R_0-1)}{R_0(b+\mu)}, \frac{\mu(R_0-1)}{R_0(b+\mu)}\right). \tag{3.4.9}$$

However, w is not part of equations 3.4.5 or 3.4.6, and hence has no effect on the spread of the infection. Thus, one need only consider u and v for the steady state analysis. Hence, the Jacobian matrix for equations 3.4.5 and 3.4.6 is given as,

$$J = \begin{bmatrix} -\frac{b}{b+\mu} - R_0 v & -R_0 u \\ R_0 v & R_0 u - 1 \end{bmatrix}, \tag{3.4.10}$$

which may then be evaluated at the steady states to determine their stability.

Evaluating the Jacobian matrix in equation 3.4.10 at the disease free steady state from equation 3.4.8, and then calculating the eigenvalues returns,

$$\lambda_1 = R_0 - 1, \tag{3.4.11}$$

$$\lambda_2 = -\frac{b}{b+\mu}. \tag{3.4.12}$$

For stability of the disease free steady state, the eigenvalues must both have negative real part. $\lambda_2 < 0$, holds since, $b > 0, \mu > 0$. However, for, $\lambda_1 < 0$, it is returned that, $R_0 < 1$, must be satisfied. If $R_0 > 1$, then the disease free steady state is unstable.

Evaluating the Jacobian matrix at the endemic steady state given in equation 3.4.9, and then calculating the eigenvalues returns,

$$\lambda_1 = -\frac{bR_0}{b+\mu}, \tag{3.4.13}$$

$$\lambda_2 = 0. \tag{3.4.14}$$

Hence it is found that the endemic steady state is always stable since $\lambda_1 < 0$. The steady state analysis is supported by the numerical solutions which follow.

Figure 11 shows that the disease will die out if R_0<1, supporting the steady state analysis. Once the infective class has become empty, so $v = 0$, the population will then become entirely susceptible to the disease again over time, as individuals in the R class will die, and new susceptibles will be born.

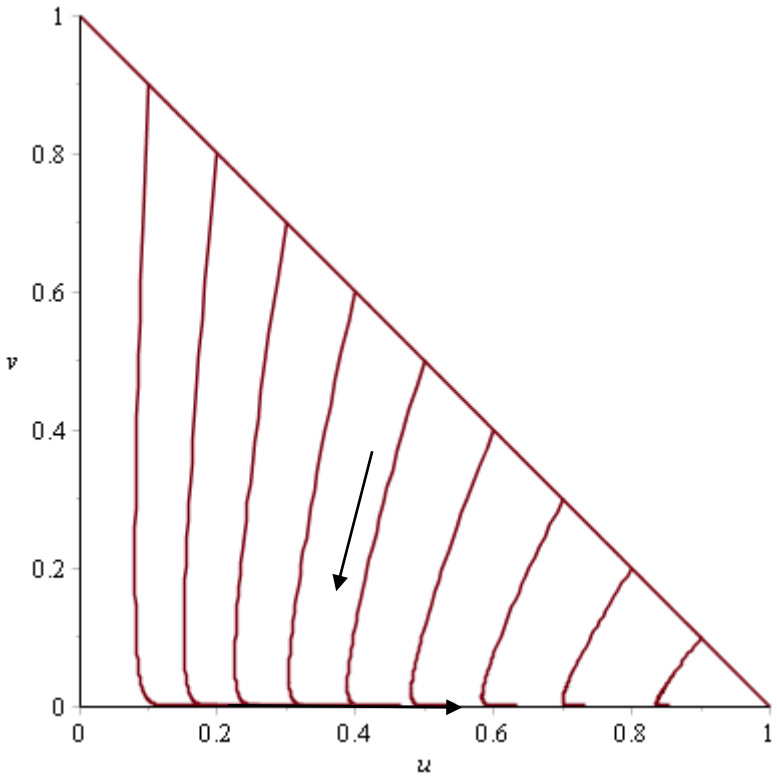

Figure 11: Phase plane for equations 3.4.5 and 3.4.6, where R_0<1.

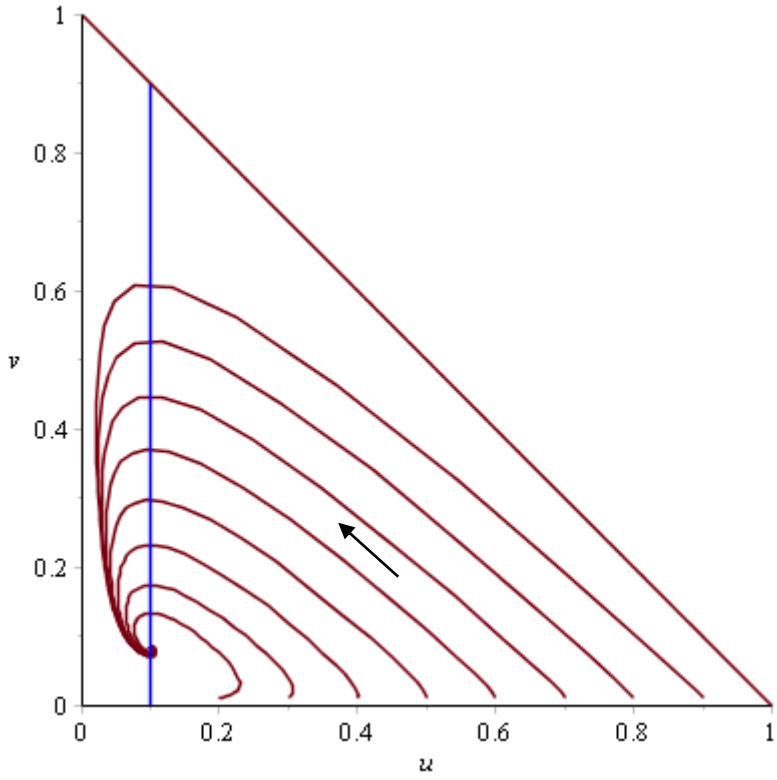

Figure 12: Phase plane for equations 3.4.5 and 3.4.6, where R_0>1.

Figure 12 shows that the disease will remain endemic in the population, provided R_0>1. This supports the steady state analysis seen previously. The steady state occurs when $u^* = \frac{1}{R_0}$. In this model, R_0 was chosen to be 10, hence $\frac{1}{R_0}$=0.1. This, supported by Figure 12, shows that after a sufficient amount of time, 10% of the population will be susceptible to the virus. From $v^* = \frac{b(R_0-1)}{R_0(b+\mu)}$, the percentage of those infected at the steady state under these modelling assumptions is calculated to be approximately 8% of the population. This is also supported by Figure 12. This gives that 82% of the population will be immune to the disease, after a sufficient amount of time has passed.

Figure 13 shows a simulation of u and v over time. Values were chosen for the parameters such that $\beta = 1$, $\mu = 10$, so that R_0=10. The initial conditions were set that $u(0) = 0.99$, $v(0) = 0.01$, meaning that when $t = 0$, 99% of the population belongs to the susceptible class, while only 1% belongs to the infective class. It shows that at the endemic steady state, approximately 10% of the population remains susceptible to the disease, with approximately 8% of the population infected.

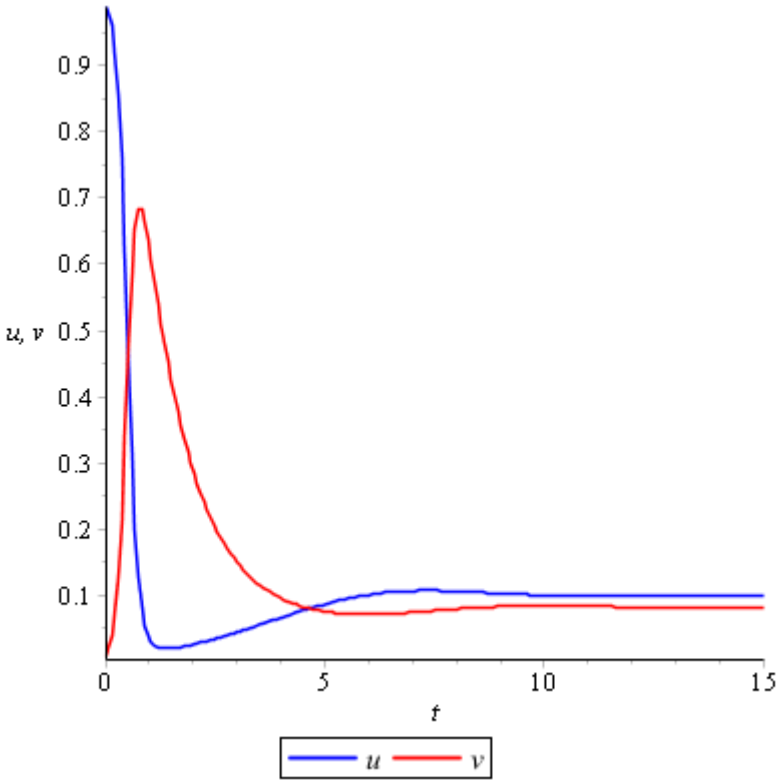

Figure 13: Plot showing the population percentages for u and v over time.

4 Modelling an Outbreak of Zombie Infection

The methods seen throughout section 3 will now be applied to an epidemic of a zombie virus. As stated in section 2, this report will model a population with the potential to become a Class 2 outbreak of zombie infection. The timescale for such an epidemic is assumed to be short, hence the following

models will see birth and death rates of the population neglected. For these models zombies take on the role of the infective class, denoted Z. In such an epidemic, it is assumed that susceptible individuals would seek to destroy zombies in order to prevent them from transmitting the virus. A second mass action transit is introduced to model this, and this addition gives the model another non-linear term, which can make analysis of the models more complicated than for the infectious disease models outlined in section 3. It will be assumed throughout this section that zombies will only infect susceptible humans, and that zombies will not destroy other zombies.

4.1 The SZR Model

The SZR model is the most basic model that will be analysed. It considers three classes of the population. Susceptibles, denoted S, are those individuals who have not yet been infected by the zombie virus, nor have they become deceased. Zombies, denoted Z, are individuals which have become zombified through infection. Finally, the removed class, denoted by R, contains individuals who have become deceased.

4.1.1 The Basic SZR Model

The Basic SZR model is outlined as follows (Smith? et al, 2009). Susceptibles may become infected, and hence become a zombie, through an interaction with a zombie. Humans in the removed class can reanimate and become a zombie. Zombies may be destroyed by a susceptible though an interaction, and on destruction zombies enter the removed class. The equations for this model are given by,

$$\frac{dS}{dt} = \Pi - \beta SZ - \delta S, \tag{4.1.1.1}$$

$$\frac{dZ}{dt} = \beta SZ - \alpha SZ + \mu R, \tag{4.1.1.2}$$

$$\frac{dR}{dt} = \delta S + \alpha SZ - \mu R. \tag{4.1.1.3}$$

Here, β is the pairwise infectious contact rate, per zombie and per susceptible. Hence the βSZ term gives the expected number of new infections per unit time. Parameter α will be referred to as the pairwise destructive contact rate, per susceptible and per zombie, since α is the rate at which a susceptible individual makes contact with a zombie sufficient to destroy that zombie by destroying the brain. Thus, the αSZ term gives the expected number of zombies that will be destroyed per unit time. The parameter δ is the natural death rate applied to the population, hence δS is the number of expected susceptible deaths from natural causes, meaning not zombie related, per unit time. Parameter μ is the rate at which an individual reanimates from the R class, and becomes a zombie once again. Finally, the birth rate in this model has been assumed to be constant, and is given by parameter Π. These equations can be represented diagrammatically as shown in figure 14.

However, equations 4.1.1.1, 4.1.1.2, and 4.1.1.3 suggest that the population is continuously growing since,

$$\frac{dS}{dt} + \frac{dZ}{dt} + \frac{dR}{dt} = \frac{dN}{dt} = \Pi, \tag{4.1.1.4}$$

which will inevitably lead to the majority of the population being zombies or dead.

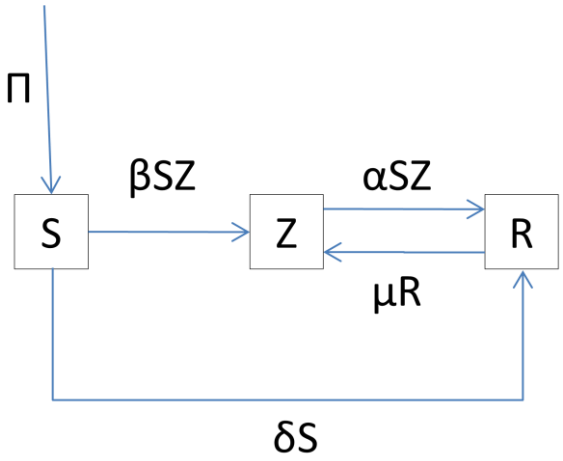

Figure 14: Flowchart for the Basic SZR model.

To rectify this, if the outbreak is assumed to happen over a short time scale, then natural birth and death rates may be neglected. Hence let, $\Pi = \delta = 0$, so that the differential equations for the SZR model become,

$$\frac{dS}{dt} = -\beta SZ, \tag{4.1.1.5}$$

$$\frac{dZ}{dt} = \beta SZ - \alpha SZ + \mu R, \tag{4.1.1.6}$$

$$\frac{dR}{dt} = \alpha SZ - \mu R. \tag{4.1.1.7}$$

Hence, now given from equations 4.1.1.5, 4.1.1.6, and 4.1.1.7 that the size of the population, N, is given by,

$$S(t) + I(t) + R(t) = N. \tag{4.1.1.8}$$

The steady states obtained from equations 4.1.1.5, 4.1.1.6, and 4.1.1.7 are,

$$(S^*, Z^*, R^*) = (N, 0, 0), \tag{4.1.1.9}$$

$$(S^*, Z^*, R^*) = (0, N, 0), \tag{4.1.1.10}$$

which will be analysed for their stability. The steady state in equation 4.1.1.9 will be referred to as the disease free steady state, since there are no zombies present in the population. However, the steady

state in equation 4.1.1.10 will be referred to as the apocalyptic steady state, since zombies have overwhelmed the susceptible population.

First, the Jacobian matrix for equations 4.1.1.5, 4.1.1.6, and 4.1.1.7 must be considered and is given by,

$$J = \begin{bmatrix} -\beta Z & -\beta S & 0 \\ (\beta - \alpha)Z & (\beta - \alpha)S & \mu \\ \alpha Z & \alpha S & -\mu \end{bmatrix}. \tag{4.1.1.11}$$

Evaluating equation 4.1.1.11 at the disease free steady state gives,

$$J(N,0,0) = \begin{bmatrix} 0 & -\beta N & 0 \\ 0 & (\beta - \alpha)N & \mu \\ 0 & \alpha N & -\mu \end{bmatrix}. \tag{4.1.1.12}$$

The eigenvalue equation obtained from equation 4.1.1.12 is,

$$\begin{aligned} \det(J - \lambda I_3) &= (-\alpha N)(-\mu\lambda) + (-\mu - \lambda)[(-\lambda)\big((\beta - \alpha)N - \lambda\big)], \\ \det(J - \lambda I_3) &= -\lambda[\lambda^2 + (\mu - (\beta - \alpha)N)\lambda - \beta\mu N], \end{aligned} \tag{4.1.1.13}$$

which is equivalent to an eigenvalue equation of the form,

$$\det(J - \lambda I_3) = \lambda^2 + a_1\lambda + a_2.$$

In equation 4.1.1.13, for the eigenvalues to have negative real part, and hence for the stability of the disease free steady state, there are some inequalities which must be satisfied. Namely,

$$a_1 > 0, \tag{4.1.1.14}$$

$$a_2 > 0, \tag{4.1.1.15}$$

where, $a_1 = (\mu - (\beta - \alpha)N)$, and $a_2 = -\beta\mu N$ (Britton, 2005). However, $a_2 < 0$, since $\beta\mu N$ is a positive number. Thus, the disease free steady state is unstable, since $\beta > 0, \mu > 0$.

Evaluating equation 4.1.1.11 at the apocalyptic steady state gives,

$$J(0,N,0) = \begin{bmatrix} -\beta N & 0 & 0 \\ (\beta - \alpha)N & 0 & \mu \\ \alpha N & 0 & -\mu \end{bmatrix}, \tag{4.1.1.16}$$

which returns the eigenvalues,

$$\lambda_1 = 0, \tag{4.1.1.17}$$

$$\lambda_2 = -\beta N, \tag{4.1.1.18}$$

$$\lambda_3 = -\mu, \tag{4.1.1.19}$$

which is sufficient for the apocalyptic steady state to be stable, since each of the eigenvalues have negative real part. Hence, it would be expected that, regardless of the values chosen for the

parameters, this model would see the zombie virus overwhelm the susceptible population, resulting in the apocalyptic steady state being reached.

A numerical simulation supports the conclusions of the analysis above. For the simulation, it was assumed that a single zombie introduced to a population of 500 susceptibles would be able to make 2.5 infectious contacts per day. Thus, $\beta = 0.005$ and hence $\beta S(0)Z(0) = 2.5$. It is also assumed that a population of 500 susceptibles would be able to destroy 3.5 zombies per day, in the presences of a single zombie. Hence, $\alpha = 0.007$ resulting in $\alpha S(0)Z(0) = 3.5$. Finally, it will be assumed that the reanimation process takes only a single day on average, hence $\mu = 1$. The simulation is shown in figure 15 and finds that under these conditions, the susceptible population is overwhelmed by the zombie horde in less than 15 days. Thus, humanity falls and the apocalyptic steady state is reached, which is the only possible outcome of a simulation for this model. Hence, this simulation shows a high level, Class 2 outbreak consisting of 501 zombies, which would have the possibility of escalating to a Class 4 if the horde moved to find a new susceptible population.

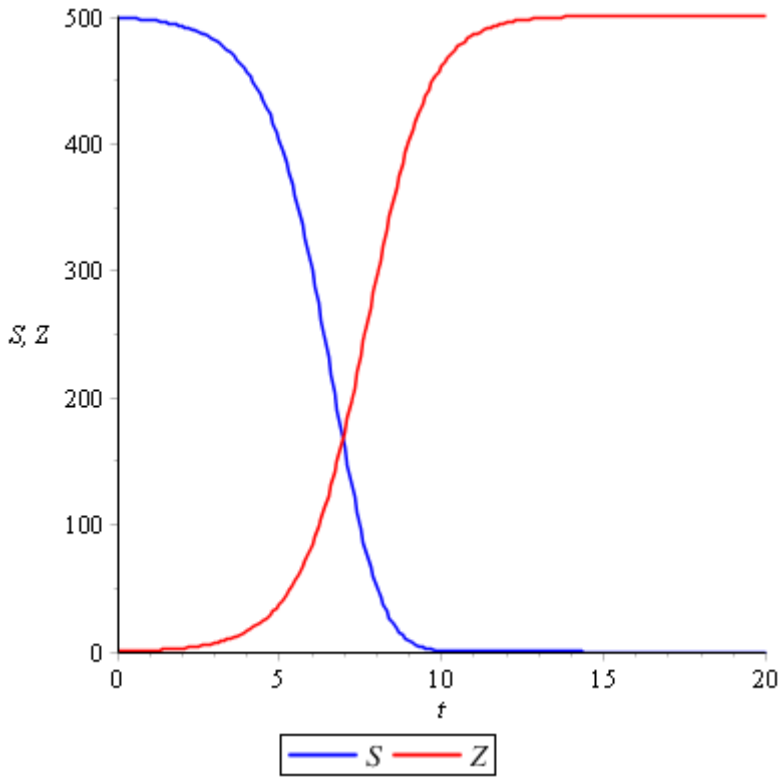

Figure 15: Simulation of the Basic SZR model.

4.1.2 An Alternative SZR Model

The Basic SZR model in section 4.1.1 makes some assumptions which may be considered to be flawed. For example, a destroyed zombie enters the removed class, and may then reanimate as a zombie again. This would suggest that the zombies possess some healing factor or regenerative

capabilities. However, as stated in section 2, this is not the case (Brooks, 2009). Hence, assume that defeated zombies simply leave the population. Thus, the R class will now consist of only deceased individuals from the susceptible class.

Zombies pass on the infection through an interaction with a susceptible. The reason a zombie interacts with a susceptible is because of their craving for human flesh, not simply to spread the infection. Hence, it is unreasonable to assume that a zombie would simply bite a susceptible individual once, and then cease the attack. Suppose instead that there is a small probability, p say, that the zombie would eat the brain of the individual during the interaction. The lack of the brain would prevent any possible zombification process or reanimation of the now deceased individual. This would result in the previously susceptible individual leaving the population since they have no further role to play in the spread of the virus. Likewise, suppose that a susceptible is wounded sufficient enough for them to become deceased, yet the brain remains whole, with a probability q. In this case, suppose the susceptible now enters the removed class from which it could reanimate as a zombie.

This model will also consider the timescale of the epidemic to be short in comparison to the life expectancy of an average human, in the absence of zombies. Hence birth and natural death rates will be neglected. Thus, this new model may be represented diagrammatically by figure 16.

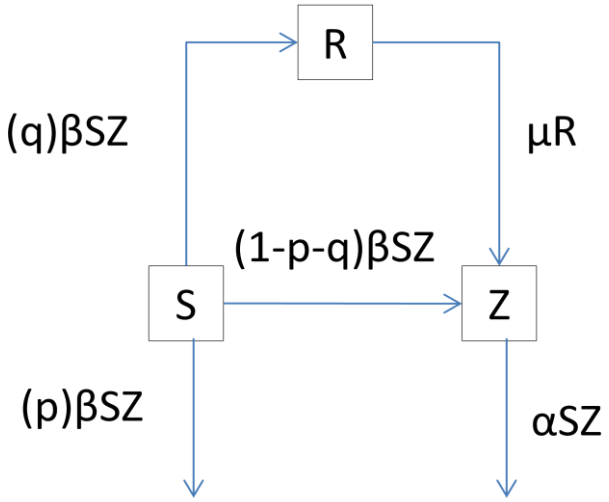

Figure 16: Flowchart of the Alternative SZR model.

Under these assumptions, the differential equations for this model are given by,

$$\frac{dS}{dt} = -\beta SZ, \tag{4.1.2.1}$$

$$\frac{dZ}{dt} = (1-p-q)\beta SZ - \alpha SZ + \mu R, \tag{4.1.2.2}$$

$$\frac{dR}{dt} = q\beta SZ - \mu R. \tag{4.1.2.3}$$

However, it is no longer retained that the population of the model, N, is given as shown in equation 4.1.1.8. Hence, N will now be defined as,

$$S(0) + Z(0) + R(0) = N, \tag{4.1.2.4}$$

where the number of individuals, D, who have left the population at time $t = \tau$ is given by,

$$N - S(\tau) - Z(\tau) - R(\tau) = D. \tag{4.1.2.5}$$

The steady states of this model, calculated from equations 4.1.2.1, 4.1.2.2, and 4.1.2.3 are given by,

$$(S^*, Z^*, R^*) = (N, 0, 0), \tag{4.1.2.6}$$

$$(S^*, Z^*, R^*) = (0, Z^*, 0), \tag{4.1.2.7}$$

which will again be analysed for their stability. The steady state in equation 4.1.2.6 is the disease free steady state, whilst equation 4.1.2.7 gives the apocalyptic steady state.

Thus, the Jacobian matrix for this system is given by,

$$J = \begin{bmatrix} -\beta Z & -\beta S & 0 \\ u(\beta - \alpha)Z & u(\beta - \alpha)S & \mu \\ q\beta Z & q\beta S & -\mu \end{bmatrix}, \tag{4.1.2.8}$$

where the substitution ,

$$u = (1 - p - q), \tag{4.1.2.9}$$

has been made for simplicity.

Evaluating the Jacobian matrix at the disease free steady state gives,

$$J(N, 0, 0) = \begin{bmatrix} 0 & -\beta N & 0 \\ 0 & u(\beta - \alpha)N & \mu \\ 0 & q\beta N & -\mu \end{bmatrix}, \tag{4.1.2.10}$$

which will be used to calculate the eigenvalues at this steady state. Hence, calculating the eigenvalue using equation 4.1.2.10 gives,

$$\det(J(N, 0, 0) - \lambda I_3) = -\lambda[(u(\beta - \alpha)N - \lambda)(-\mu - \lambda) - (q\beta\mu N)],$$

$$\det(J(N, 0, 0) - \lambda I_3) = -\lambda[\lambda^2 + (\mu - u(\beta - \alpha)N)\lambda + (-u(\beta - \alpha) - q\beta)\mu N]. \tag{4.1.2.11}$$

Thus for stability, the inequality,

$$a_2 = (-u(\beta - \alpha) - q\beta)\mu N > 0, \tag{4.1.2.12}$$

must be satisfied. This may be rearranged to give,

$$\alpha > \frac{1 - p}{1 - p - q}\beta. \tag{4.1.2.13}$$

Thus, provided equations 4.1.2.13 and 4.1.1.14 are satisfied, the disease free steady state will be stable, meaning that humanity could survive the zombie outbreak. However,

$$a_1 = (\mu - u(\beta - \alpha)N) > 0,$$

must also hold, and hence it is returned that the inequality,

$$\alpha > \beta - \frac{1}{uN}, \tag{4.1.2.14}$$

must be satisfied.

However, the Jacobian matrix must also be evaluated at the apocalyptic steady state to determine its stability. This gives,

$$J(0, Z^*, 0) = \begin{bmatrix} -\beta Z^* & 0 & 0 \\ u(\beta - \alpha)Z^* & 0 & \mu \\ q\beta Z^* & 0 & -\mu \end{bmatrix}, \tag{4.1.2.15}$$

which returns eigenvalues,

$$\lambda_1 = 0,$$
$$\lambda_2 = -\mu,$$
$$\lambda_3 = -\beta Z^*.$$

Thus the apocalyptic steady state is always stable. Numerical simulations of the model support the stability analysis.

The simulation shown in figure 17 shows the disease free steady state is stable. In this simulation, it was assumed that a population consisting of 500 susceptible individuals, and just a single zombie, would be capable of destroying 3.5 zombies per unit time. Thus giving that $\alpha S(0)Z(0) = 3.5$ and hence $\alpha = 0.007$. As in section 4.1.1, it has been assumed that a single zombie, introduced into a population of 500 susceptible individuals would be able to make 2.5 contacts sufficient to transmit the infection per unit time. Thus, as in section 4.1.1, giving that $\beta = 0.005$. Likewise, the parameter $\mu = 1$ was also retained from section 4.1.1. It is also assumed that when a zombie makes contact with a susceptible, that there is a probability of 0.05 that the zombie consumes the brain of the individual, thus $p = 0.05$, and that there is a probability of 0.1 that the susceptible individual is killed from their would, hence $q = 0.1$.These values for the parameters give that the inequality in equation 4.1.2.13 is satisfied, along with the inequality from equation 4.1.2.14, thus satisfying the condition for the stability of the disease free steady state. Figure 17 shows that, with a small number of susceptible fatalities, the zombie horde is eradicated in approximately 15 days. Thus, only a Class 1 outbreak occurred in this simulation, which was eradicated with minimal losses to human life.

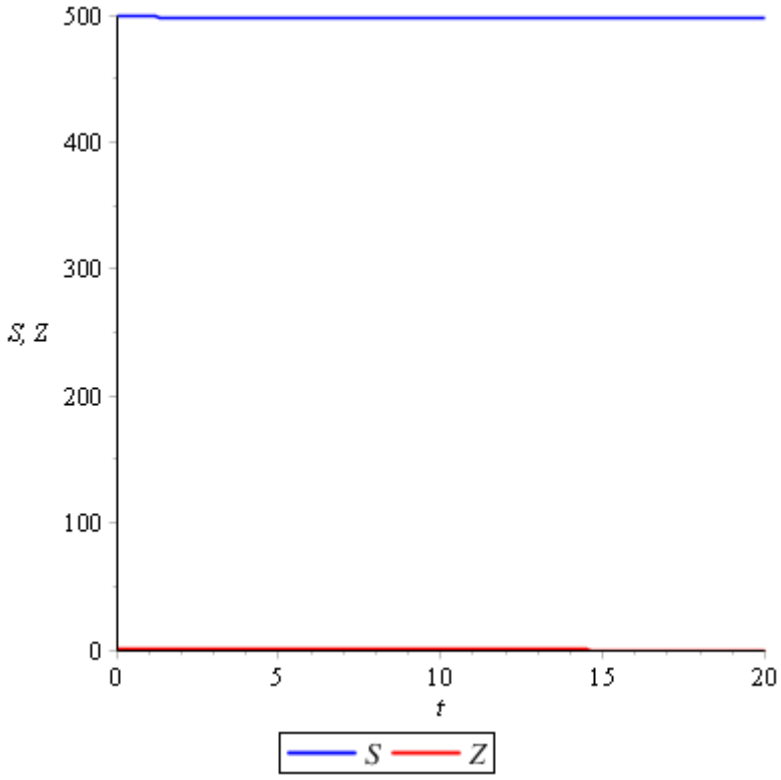

Figure 17: Simulation of the Alternative SZR model with stability of the disease free steady state.

The simulation in figure 18 makes a different assumption. Here, it is assumed that a population of 500 susceptibles, with just a single zombie present, would be able to kill only 1.5 zombies per day. Hence giving that $\alpha S(0)Z(0) = 1.5$, and thus $\alpha = 0.003$. All other values for parameters are retained from the simulation in figure 17. Hence, the inequality in equation 4.1.2.13 is no longer satisfied, which causes the disease free steady state to become unstable. Thus, figure 18 shows that the zombies quickly overwhelm the population in approximately 25 days. After this point, individuals in the removed class would reanimate as zombies, until only the zombie class remains populated. However, the survival time in this simulation is an improvement on the survival time in the simulation in figure 15 in section 4.1.1, even with a decreased value of α. This is due to the decreased number of zombies present in the model, since a zombie, once defeated, leaves the population and is unable to reanimate, and since there is a probability of 0.05 that a susceptible individual is removed from the population in an encounter with a zombie, hence never becoming a zombie. This simulation shows a low level Class 2 outbreak, with total loss of life in the susceptible population, but only approximately 200 zombies are created. However, even a low level, Class 2 outbreak presents a significant threat, and could still become a Class 4 outbreak if left unchecked.

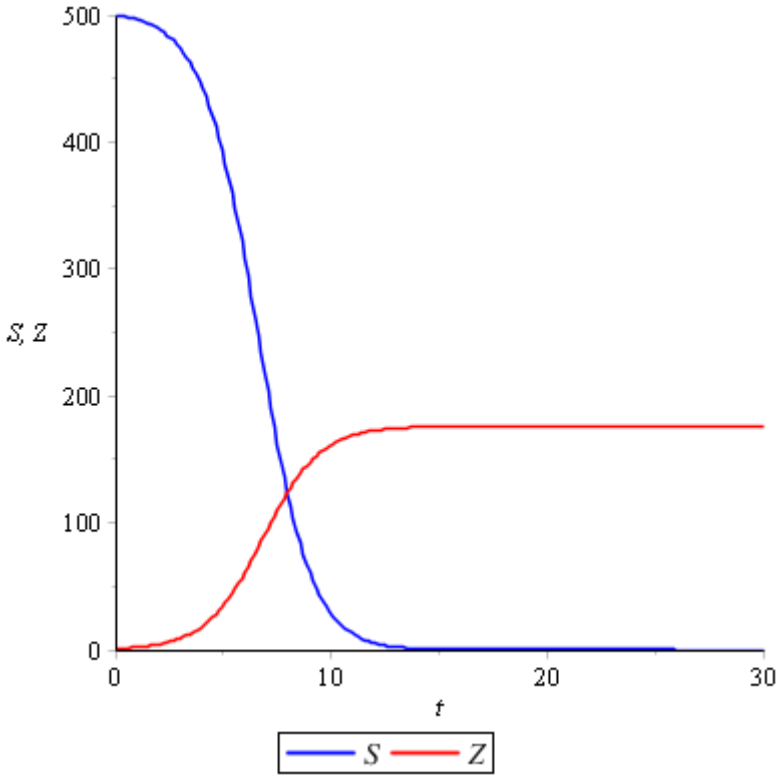

Figure 18: Simulation of the Alternative SZR model with unstable disease free steady state.

This model is more beneficial to the survival of humanity than the model outlined in section 4.1.1, since it takes more time for humanity to be overwhelmed under conditions where the apocalyptic steady state is stable. This model also allows the disease free steady state to be stable, which was not possible in the previous model. Thus this model gives the population the possibility of survival, provided the susceptible population is able to destroy zombies at a sufficient rate, which would prevent the Class 2 outbreak from escalating to a higher class of outbreak.

4.2 The SLZR Model

This model introduces the idea that there is a latent period of infection, where the individual has been infected by the zombie virus, but has not yet become a zombie (Brooks, 2009: Britton, 2005). Suppose now that zombification occurs at some rate, ρ, after a susceptible individual has become infected with the virus. It has been suggested that there is on average a 24 hour period after infection, before zombification of the individual takes place. This is supported by other sources where an individual, begins to show signs of sickness after being bitten, but before they turn into a zombie (Telltale Games, 2012). Hence the model is modified so that a susceptible enters a latent class, L, before moving on to become a zombie and entering the Z class, at some rate ρ.

4.2.1 The Basic SLZR Model

A Basic model for an outbreak of zombie infection with a latent infective class is outlined as follows (Smith? et al, 2009). As stated in section 4.2, it will be assumed that upon infection, an individual moves from the susceptible class, S, to the latent class, L, and then, at some rate ρ, moves to the zombie class, Z. It has again been assumed that natural birth and death rates may be neglected, since the timescale for the model is expected to be short in comparison to the life expectancy of the average human. Thus, the differential equations which S, L, Z and R satisfy for this model are given by,

$$\frac{dS}{dt} = -\beta SZ, \tag{4.2.1.1}$$

$$\frac{dL}{dt} = \beta SZ - \rho L, \tag{4.2.1.2}$$

$$\frac{dZ}{dt} = \rho L - \alpha SZ + \mu R, \tag{4.2.1.3}$$

$$\frac{dR}{dt} = \alpha SZ - \mu R. \tag{4.2.1.4}$$

The differential equations may be represented diagrammatically as given in figure 19.

The steady states of the differential equations in equations 4.2.1.1, 4.2.1.2, 4.2.1.3, and 4.2.1.4 are,

$$(S^*, L^*, Z^*, R^*) = (N, 0, 0, 0), \tag{4.2.1.5}$$

$$(S^*, L^*, Z^*, R^*) = (0, 0, N, 0). \tag{4.2.1.6}$$

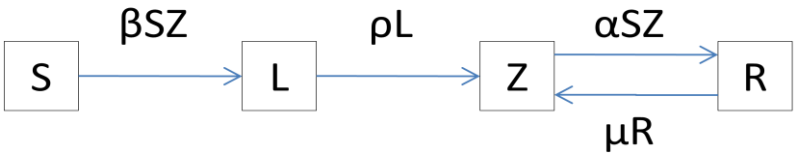

Figure 19: Flowchart for the Basic SLZR model.

Equation 4.2.1.5 gives the disease free steady state, while equation 4.2.1.6 gives the apocalyptic steady state. In order to determine the stability of these steady states, the Jacobian matrix will be determined.

$$J = \begin{bmatrix} -\beta Z & 0 & -\beta S & 0 \\ \beta Z & -\rho & \beta S & 0 \\ -\alpha Z & \rho & -\alpha S & \mu \\ \alpha Z & 0 & \alpha S & -\mu \end{bmatrix}, \tag{4.2.1.7}$$

which can be evaluated at the disease free steady state given in equation 4.2.1.5 to give,

$$J(N,0,0,0) = \begin{bmatrix} 0 & 0 & -\beta N & 0 \\ 0 & -\rho & \beta N & 0 \\ 0 & \rho & -\alpha N & \mu \\ 0 & 0 & \alpha N & -\mu \end{bmatrix}. \tag{4.2.1.8}$$

Thus, to determine the eigenvalues at the disease free steady state, the eigenvalue equation is calculated as,

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda\left[\det\left(\begin{bmatrix} -\rho-\lambda & \beta N & 0 \\ \rho & -\alpha N-\lambda & \mu \\ 0 & \alpha N & -\mu-\lambda \end{bmatrix}\right)\right],$$

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda[(-\rho-\lambda)[(-\alpha N-\lambda)(-\mu-\lambda) - \mu\alpha N] - (\beta N)[\rho(-\mu-\lambda)]],$$

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda(-\lambda^3 - a_1\lambda^2 - a_2\lambda - a_3), \tag{4.2.1.9}$$

where the substitutions,

$$a_1 = (\rho + \mu + \alpha N),$$
$$a_2 = (\mu\rho + \alpha\rho N - \beta\rho N),$$
$$a_3 = -\beta\mu\rho N,$$

have been made. For the stability of an eigenvalue equation which is of the form,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

the inequalities,

$$a_1 > 0, \tag{4.2.1.10}$$
$$a_3 > 0, \tag{4.2.1.11}$$
$$a_1 a_2 - a_3 > 0, \tag{4.2.1.12}$$

must be satisfied (Britton, 2005). Thus, these inequalities are also retained for an eigenvalue equation of the form,

$$-\lambda^3 - a_1\lambda^2 - a_2\lambda - a_3 = 0.$$

However, $a_3 = -\beta\mu\rho N < 0$, since $\beta > 0, \mu > 0, \rho > 0, N > 0$. Thus the inequality in equation 4.2.1.11 is not satisfied, hence the disease free steady state is always unstable for this model. Hence, it is expected that no simulation of this model would return a situation where the population has overcome the zombie horde.

Evaluating the Jacobian matrix from equation 4.2.1.7 at the apocalyptic steady state given in equation 4.2.1.6 gives,

$$J(0,0,Z^*,0) = \begin{bmatrix} -\beta Z^* & 0 & 0 & 0 \\ \beta Z^* & -\rho & 0 & 0 \\ -\alpha Z^* & \rho & 0 & \mu \\ \alpha Z^* & 0 & 0 & -\mu \end{bmatrix}. \tag{4.2.1.13}$$

The eigenvalues calculated from equation 4.2.1.13 are given as,

$$\lambda_1 = 0,$$
$$\lambda_2 = -\rho,$$
$$\lambda_3 = -\mu,$$
$$\lambda_4 = -\beta Z^*.$$

Thus, the apocalyptic steady state will always be stable, since all the eigenvalues have negative real part. Hence, it would be expected for zombies to overwhelm the susceptible population, regardless of the values chosen for the parameters.

A numerical simulation of this model supports the analysis. As with the previous simulation in section 4.1.1, it has been assumed that $\alpha S(0)Z(0) = 3.5$, and that $\beta S(0)Z(0) = 2.5$, along with $\mu = 1$. Given that an individual is assumed to spend, on average, 24 hours in the latent class, $\rho = 1$ is used, since the simulation is measuring the time units in days. In order to conform with the assumptions for a Class 2 outbreak of zombie infection outlined in section 2, the initial conditions of the population are $S(0) = 500$, $Z(0) = 1$, and $L(0) = R(0) = 0$. This meant that the simulation started with 500 susceptibles and just a single zombie, with none of the population starting in the removed or latent class. Figure 20 shows that the addition of the latent class increases the time it takes for the zombies to infect all of the susceptible population when compared with figure 15 in section 4.1.1. Here it takes almost 20 days for humanity to fall, whilst previously it took less than 15. So the delay in new zombies entering the population, provided by the addition of the latent class, allows the susceptible population to survive for a slightly longer time period. However, it is still found that humanity falls and that the zombies will take over, thus this model is unable to prevent the high level Class 2 outbreak from overwhelming the susceptible population, which may result in the outbreak escalating to a Class 4.

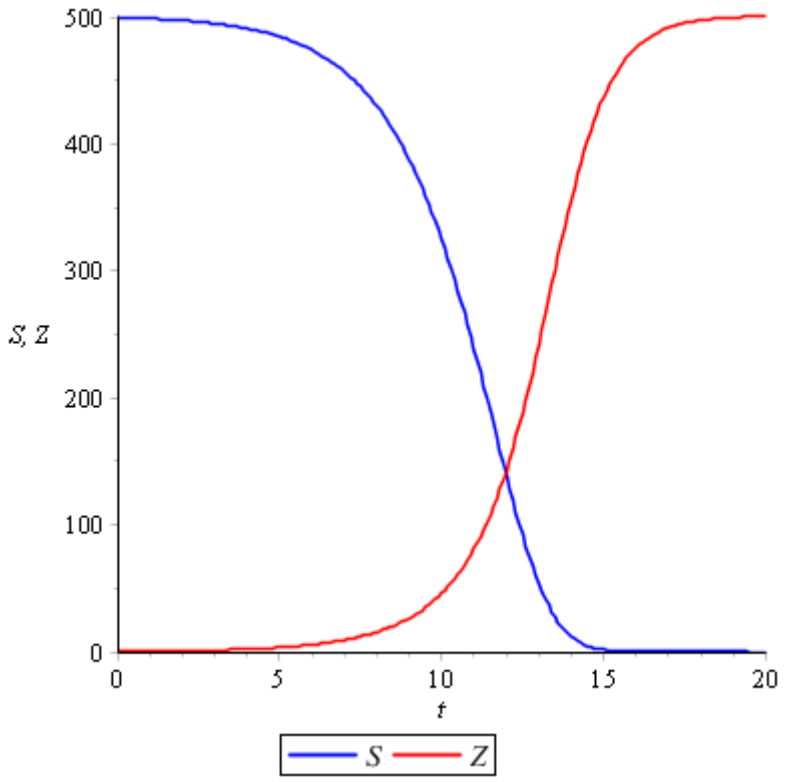

Figure 20: Simulation of the SLZR model for zombie infection.

4.2.2 An Alternative SLZR Model

As the model for latent infection in section 4.2.1 is an extension of the Basic SZR model given in section 4.1.1, the Alternative SZR model from section 4.1.2 will now have a latent class introduced in order to modify the assumptions made in the Basic SLZR model. The latent class will have the same properties as the in section 4.2.1 in that an individual enters the latent class upon infection, rather than immediately entering the zombie class, and individuals will leave the latent class and enter the zombie class at a rate ρ. Again, the average time spent in the latent class will be assumed to be twenty four hours. The probability of what happens to an individual upon coming into contact with a zombie will also be retained along with the substitution for u given in equation 4.1.2.9. Thus, the differential equations for this model are,

$$\frac{dS}{dt} = -\beta SZ, \tag{4.2.2.1}$$

$$\frac{dL}{dt} = u\beta SZ - \rho L, \tag{4.2.2.2}$$

$$\frac{dZ}{dt} = \rho L - \alpha SZ + \mu R, \tag{4.2.2.3}$$

$$\frac{dR}{dt} = q\beta SZ - \mu R. \tag{4.2.2.4}$$

This model is also represented diagrammatically in figure 21.

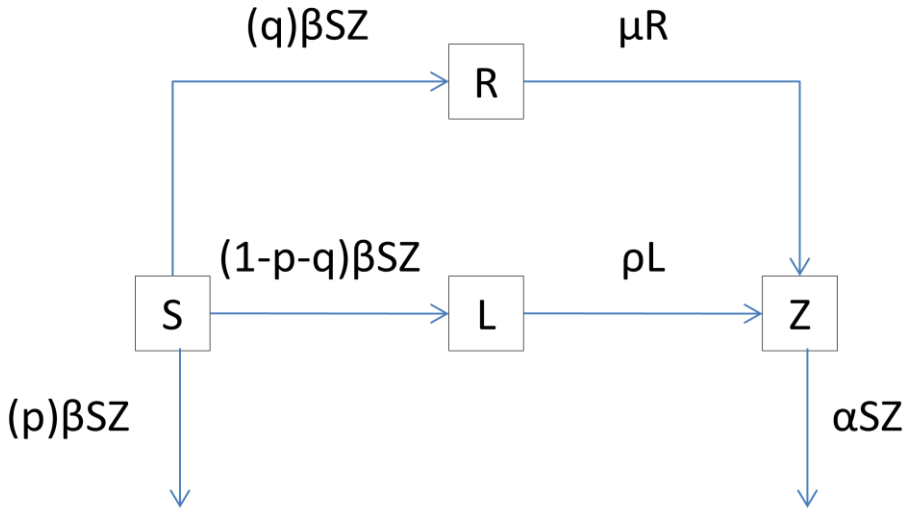

Figure 21: Flowchart for the Alternative SLZR model.

However, despite having some alterations made to the assumptions of the model, the steady states are equal to those in equations 4.2.1.5 and 4.2.1.6. Hence, the Jacobian matrix will again be calculated and then evaluated at the steady states in order to determine if humanity may survive under these alternate modelling assumptions and thus preventing a Class 2 outbreak from escalating any further. Thus, the Jacobian matrix is now,

$$J = \begin{bmatrix} -\beta Z & 0 & -\beta S & 0 \\ u\beta Z & -\rho & u\beta S & 0 \\ -\alpha Z & \rho & -\alpha S & \mu \\ q\beta Z & 0 & q\beta S & -\mu \end{bmatrix}, \tag{4.2.2.5}$$

which, when evaluated at the disease free steady state given in equation 4.2.1.5 becomes,

$$J(N,0,0,0) = \begin{bmatrix} 0 & 0 & -\beta N & 0 \\ 0 & -\rho & u\beta N & 0 \\ 0 & \rho & -\alpha N & \mu \\ 0 & 0 & q\beta N & -\mu \end{bmatrix}. \tag{4.2.2.6}$$

Next, the eigenvalue equation is calculated,

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda\left[\det\left(\begin{bmatrix} -\rho-\lambda & u\beta N & 0 \\ \rho & -\alpha N-\lambda & \mu \\ 0 & q\beta N & -\mu-\lambda \end{bmatrix}\right)\right],$$

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda[(-\rho-\lambda)[(-\alpha N-\lambda)(-\mu-\lambda) - q\beta\mu N] - (u\beta N)[\rho(-\mu-\lambda)]],$$

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda(-\lambda^3 - a_1\lambda^2 - a_2\lambda - a_3), \tag{4.2.2.7}$$

where,

$$a_1 = (\rho + \mu + \alpha N), \tag{4.2.2.8}$$

$$a_2 = (\mu\rho + \alpha\rho N + \alpha\mu N - q\beta\mu N - u\beta\rho N), \quad (4.2.2.9)$$

$$a_3 = \alpha\mu\rho N - (u+q)\beta\mu\rho N. \quad (4.2.2.10)$$

As in section 4.2.1, to determine the stability of this steady state, the inequalities given in equations 4.2.1.10, 4.2.1.11, and 4.2.1.12 must be satisfied. Thus, from equation 4.2.2.10,

$$a_3 = \alpha\mu\rho N - (u+q)\beta\mu\rho N > 0,$$

and hence,

$$\alpha > (u+q)\beta = (1-p)\beta, \quad (4.2.2.11)$$

must be satisfied in order for the disease free steady state to be stable. It is also required that equation 4.2.1.12 is satisfied. However, this inequality will be shown to be satisfied on choosing values for the parameters in the simulations. Equation 4.2.1.10 is satisfied since $\alpha, \mu, \rho > 0$. Thus, for now it will be assumed that, provided that the susceptible population is able to destroy the zombies at a rate α which is sufficiently large enough to satisfy the inequality in equation 4.2.2.11, the disease free steady state will be stable, and humanity may survive the zombie apocalypse.

Now evaluating the Jacobian matrix from equation 4.2.2.5 at the apocalyptic steady state given in equation 4.2.1.6, the Jacobian matrix becomes,

$$J(0,0,Z^*,0) = \begin{bmatrix} -\beta Z^* & 0 & 0 & 0 \\ 0.85\beta Z^* & -\rho & 0 & 0 \\ -\alpha Z^* & \rho & 0 & \mu \\ 0.1\beta Z^* & 0 & 0 & -\mu \end{bmatrix}. \quad (4.2.2.12)$$

The eigenvalues for equation 4.2.2.12 are given as,

$$\lambda_1 = -\beta Z^*, \lambda_2 = -\rho, \lambda_3 = 0, \lambda_4 = -\mu,$$

and since $\beta, \rho, \mu, Z^* > 0$, all the eigenvalues will have negative real part, hence the apocalyptic steady state is always stable. Hence, it is assumed that when the conditions for the stability of the disease free steady state are not satisfied, the apocalyptic steady state will be reached. Thus, if humanity is not able to kill zombies at a sufficient rate, they will be overwhelmed and the outbreak may escalate beyond that of a Class 2. Once again, numerical simulations of this model support the analysis of the steady states.

The parameters in the simulation show by figure 22 were chosen to be the same as those for the simulation shown in figure 17 since the same assumptions are being made about the population with parameter ρ obtained from the simulation shown in figure 20. These choices satisfy the inequalities in equations 4.2.1.11 and 4.2.1.12 using equations 4.2.2.8, 4.2.2.9, and 4.2.2.10, with the inequality in equation 4.2.1.12 becoming, $32.06 > 0$, which holds and is hence sufficient for the stability of the disease free steady state. The initial conditions on the population were $S(0) = 500, L(0) = 0, Z(0) = 1, R(0) = 0$, so that once again, there is the potential for the outbreak to become a Class 2 outbreak.

This simulation shows that the susceptible population is able to eradicate the zombie horde in approximately 34 days, with only a small number of fatalities, thus preventing the outbreak from escalating beyond that of a Class 1. Comparing this with the simulation in figure 17 for the Alternative SZR model finds that the introduction of a latent class prolongs the length of the outbreak, since there is now a delay on an individual entering the zombie class after infection.

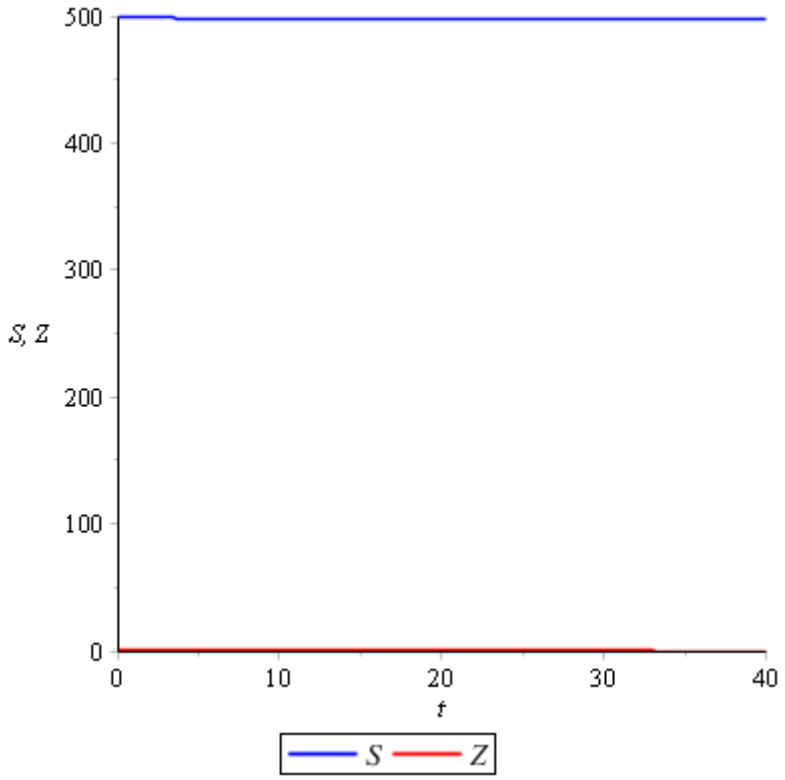

Figure 22: Simulation of the Alternative SLZR model with stable disease free steady state.

The simulation in figure 23 uses the same values for the parameters and initial conditions as the simulation in figure 22, with the exception that $\alpha = 0.003$ by the same assumptions made in the simulation shown in figure 18 for the rate at which the susceptible population is able to destroy zombies. This choice for α no longer satisfies the inequality in equation 4.2.1.11, resulting in the instability of the disease free steady state. The simulation also shows that the time taken for humanity to fall is approximately 35 days, which is a longer time than in all previous apocalyptic simulations considered in sections 4.1.1, 4.1.2, and 4.2.1. However, despite the extinction of humanity, there is a significant decrease in the zombie population at the steady state of this model, compared to the zombie population at the apocalyptic steady state shown in figure 15 and figure 20. Thus, the outbreak is not as severe at the steady state, meaning that while it may escalate beyond a low level Class 2

outbreak, there are fewer zombies which must be dealt with at this point than in the simulation shown in figure 20, and hence the outbreak would be potentially easier to eradicate by some external force.

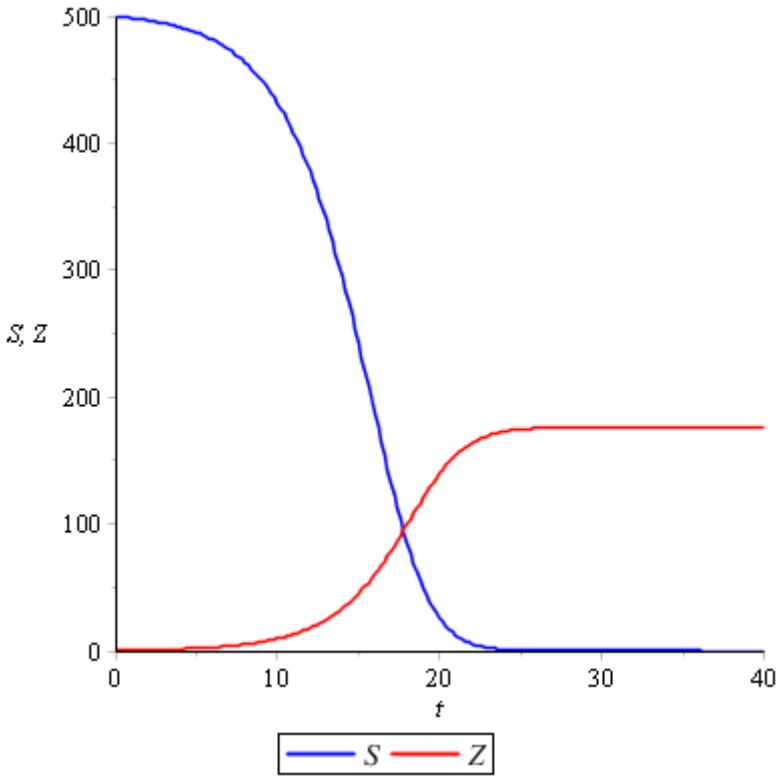

Figure 23: Simulation of the Alternative SLZR model with unstable disease free steady state.

4.3 The SLZR Model with a cure

Suppose that it were possible to produce a cure to counteract the zombie virus. This model will assume that a cure is readily available at the start of the outbreak, thus requiring no research or development time. However, it will also be assumed that the cure provides no immunity against the virus, and that there is no time delay for the cure to take effect. Hence, upon application of the cure, an individual would return to the susceptible class, and may again be infected by a zombie.

4.3.1 The Basic SLZR Model with a cure

The basic model for an SLZR model with a cure for the zombie virus is outlined as follows (Smith? et al, 2009). It is proposed that the cure is applied to individuals in the zombie class, at some rate given by parameter c. Upon being cured, the individual enters the susceptible class, where it is again susceptible to infection. The differential equations to model this are as follows,

$$\frac{dS}{dt} = -\beta SZ + cZ, \tag{4.3.1.1}$$

$$\frac{dL}{dt} = \beta SZ - \rho L, \tag{4.3.1.2}$$

$$\frac{dZ}{dt} = \rho L + \mu R - \alpha SZ - cZ, \tag{4.3.1.3}$$

$$\frac{dR}{dt} = \alpha SZ - \mu R. \tag{4.3.1.4}$$

This can be represented diagrammatically as shown in figure 24.

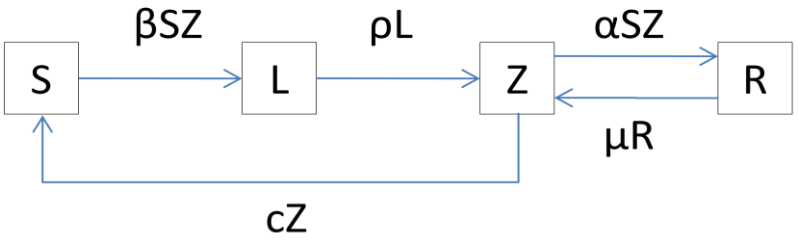

Figure 24: Flowchart for the Basic SLZR model with a cure for the zombie virus.

Under these modelling assumptions, the disease free steady state, from equation 4.2.1.5 is obtained. However, there is now a steady state in which the susceptible population and the zombies may coexist. This steady state is given by,

$$(S^*, L^*, Z^*, R^*) = \left(\frac{c}{\beta}, \frac{cZ^*}{\rho}, Z^*, \frac{\alpha c Z^*}{\beta\mu}\right). \tag{4.3.1.5}$$

Once again, using the Jacobian matrix, the stability of the disease free steady state will be determined, since this is the preferable outcome from the zombie outbreak. Thus, the Jacobian matrix for this model is given by,

$$J = \begin{bmatrix} -\beta Z & 0 & -\beta S + c & 0 \\ \beta Z & -\rho & \beta S & 0 \\ -\alpha Z & \rho & -\alpha S - c & \mu \\ \alpha Z & 0 & \alpha S & -\mu \end{bmatrix}. \tag{4.3.1.6}$$

Next, evaluating the Jacobian matrix from equation 4.3.1.6 at the disease free steady state given in equation 4.2.1.5 gives,

$$J(N,0,0,0) = \begin{bmatrix} 0 & 0 & -\beta N + c & 0 \\ 0 & -\rho & \beta N & 0 \\ 0 & \rho & -\alpha N - c & \mu \\ 0 & 0 & \alpha N & -\mu \end{bmatrix}, \tag{4.3.1.7}$$

which gives the eigenvalue equation as,

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda\left[\det\left(\begin{bmatrix} -\rho - \lambda & \beta N & 0 \\ \rho & -\alpha N - c - \lambda & \mu \\ 0 & \alpha N & -\mu - \lambda \end{bmatrix}\right)\right],$$

$$\det(J(N,0,0,0)-\lambda I_4)=-\lambda[(-\rho-\lambda)[(-\alpha N-c-\lambda)(-\mu-\lambda)-\mu\alpha N]-(\beta N)[\rho(-\mu-\lambda)]],$$

$$\det(J(N,0,0,0)-\lambda I_4)=-\lambda(-\lambda^3-a_1\lambda^2-a_2\lambda-a_3). \tag{4.3.1.8}$$

In equation 4.3.1.8, the coefficients a_1, a_2, a_3, are given as,

$$a_1=(c+\rho+\mu+\alpha N), \tag{4.3.1.9}$$

$$a_2=(\mu\rho+\alpha\rho N+c\mu-\beta\rho N), \tag{4.3.1.10}$$

$$a_3=c\mu\rho-\beta\mu\rho N. \tag{4.3.1.11}$$

Thus, for the stability of the disease free steady state, the inequalities in equations 4.2.1.10, 4.2.1.11, and 4.2.1.12 must be satisfied. Hence, from equations 4.2.1.11 and 4.3.1.11, it is found that,

$$\beta N<c, \tag{4.3.1.12}$$

must be satisfied in order for the disease free steady state to be stable. Hence, provided that zombies are being cured at a rate which is faster than the rate at which susceptibles are infected, the disease free steady state will be stable. However, the inequality in equation 4.2.1.12 must also be satisfied. The number of terms in this inequality makes the analysis difficult. So upon choosing numerical values for the parameters, this inequality will evaluated to determine if it is satisfied.

Next, the coexistence steady state will be analysed. Evaluating at this steady state, the Jacobian matrix is given by,

$$J\left(\frac{c}{\beta},\frac{cZ^*}{\rho},Z^*,\frac{\alpha cZ^*}{\beta\mu}\right)=\begin{bmatrix}-\beta Z^* & 0 & 0 & 0\\ \beta Z^* & -\rho & c & 0\\ -\alpha Z^* & \rho & -m-c & \mu\\ \alpha Z^* & 0 & m & -\mu\end{bmatrix}, \tag{4.3.1.13}$$

where, $m=\frac{\alpha c}{\beta}$. The eigenvalue equation is then given by,

$$\det\left(J\left(\frac{c}{\beta},\frac{cZ^*}{\rho},Z^*,\frac{\alpha cZ^*}{\beta\mu}\right)-\lambda I_4\right)=-\lambda\left[\det\left(\begin{bmatrix}-\rho-\lambda & c & 0\\ \rho & -m-c-\lambda & \mu\\ 0 & m & -\mu-\lambda\end{bmatrix}\right)\right],$$

$$\det(J(N,0,0,0)-\lambda I_4)=-\lambda[(-\rho-\lambda)[(-m-c-\lambda)(-\mu-\lambda)-\mu m]-(c)[\rho(-\mu-\lambda)]],$$

$$\det(J(N,0,0,0)-\lambda I_4)=-\lambda(\lambda^2+a_1\lambda+a_2), \tag{4.3.1.14}$$

with the coefficients a_1, a_2, given by,

$$a_1=(c+\rho+\mu+m), \tag{4.3.1.15}$$

$$a_2=(\mu\rho+\rho m+2c\rho+c\mu). \tag{4.3.1.16}$$

Hence the coexistence steady state is always stable since $a_1>0, a_2>0,$ as, $c>0, \rho>0, \mu>0, m>0$. Thus equations 4.1.1.14 and 4.1.1.15 are satisfied. Hence, if the zombies are not being cured at a sufficient rate, the coexistence steady state given by equation 4.3.1.5 will be reached.

Figure 25 shows a simulation of the Basic SLZR model with a cure for the zombie virus. The assumptions made about the rates at which the susceptible population destroys zombies, and zombies infect susceptibles are retained from the simulation shown in figure 20 in section 4.1. Thus, it has again been chosen that $\alpha = 0.007, \beta = 0.005$. The assumptions regarding the reanimation rate, μ, and the rate at which an individual in the latent class becomes a zombie, parameter ρ, are also retained. Hence, the substitution, $\mu = \rho = 1$, has been made. The new assumption made in this model is that it would take, on average, only one third of a day for a zombie to be cured of the virus. Hence, it was chosen that $c = 3$. Thus, the conditions for stability of the disease free steady state, given in equation 4.3.1.12, were satisfied, with the inequality in equation 4.2.1.12 becoming, $42 > 0$, using equations 4.3.1.9, 4.3.1.10 and 4.3.1.11. Thus, this simulation shows that the zombie virus is eradicated immediately, with no losses to susceptible life, and hence suggesting that this model would be able to prevent a Class 2 outbreak, since this simulation never escalates above that of a Class 1 outbreak.

Figure 26 shows a simulation of the Basic SLZR model with a cure for the zombie virus, where the coexistence steady state is reached. Whilst the assumptions regarding parameters β, μ and ρ are the same as in the simulation shown in figure 25, the assumptions regarding parameters α and c have changed. Here it is assumed that the rate at which the susceptible population destroys zombies is the same as for the simulation shown in figure 18 in section 4.1.2. However, it has now been assumed that on average it would take a zombie 200 days to be cured of the virus. Hence it was chosen that $c = 0.005$. Under these assumptions, the inequality in equation 4.3.1.12 is unsatisfied, hence the disease free steady state is no longer stable. The simulation shows that it takes approximately 20 days for the coexistence steady state to be reached. Figure 26 also shows that the susceptible population only exists in low numbers once the steady state has been reached. Although this may be considered to be better than the extinction of humanity, it is not a desirable outcome since it depends on zombies being cured each day, while members of the susceptible population are still being infected. However, whilst the zombie horde is not eradicated under these modelling assumptions, it is also held constant provided there is no limit on the supply of the cure, thus preventing the Class 2 outbreak from escalating to a higher class.

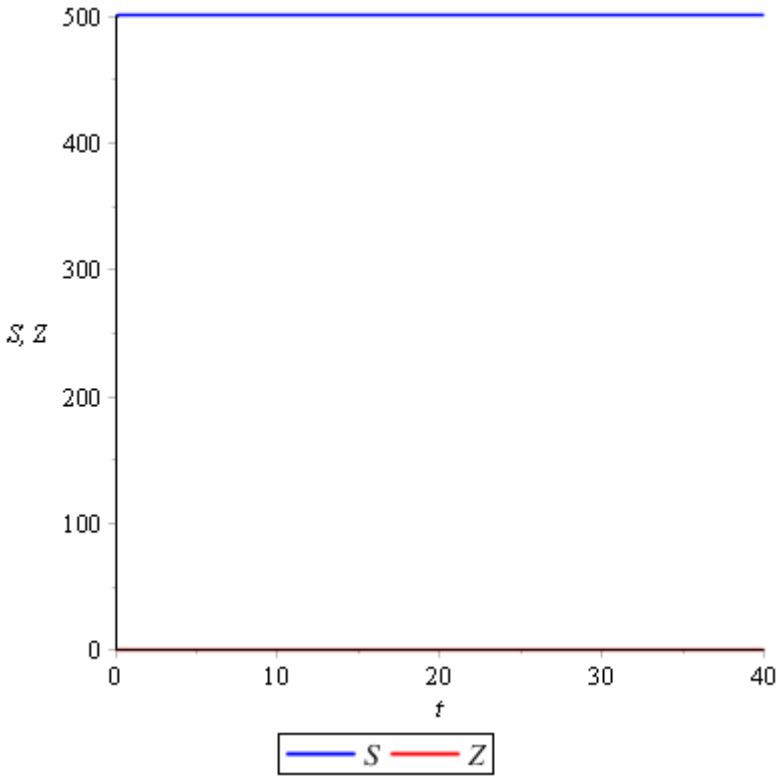

Figure 25: Simulation of the Basic SLZR model with a cure with stable disease free steady state.

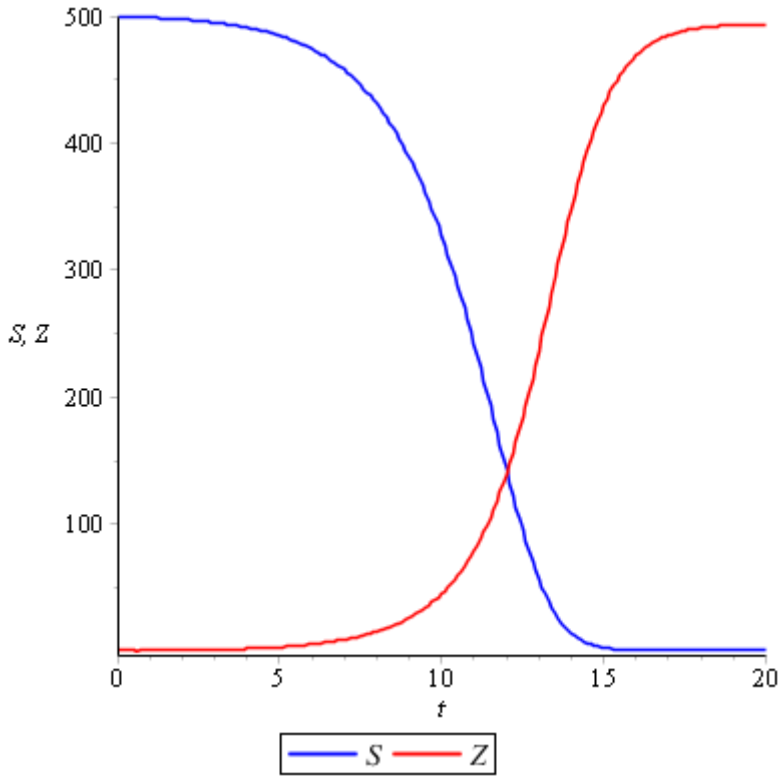

Figure 26: Simulation of the Basic SLZR model with a cure, where the coexistence steady state is stable.

4.3.2 An Alternative SLZR model with a Counter-Virus

In section 4.3.2, it had been assumed that the cure would be applied to zombies. However, given the definition that a zombie is a reanimated human corpse (Brooks, 2009), the human must have died at some point prior to zombification. Hence, a cure being applied in this manner implies that it can also cure death. There is no such cure in existence. Hence, it would be much more reasonable to assume the existence of a counter-virus as opposed to a cure (Brooks, 2009). Therefore, this model will apply the counter-virus to individuals in the latent class, at a rate given by parameter c. As before, the counter-virus will convey no immunity against the zombie virus. Once the counter-virus is applied to an individual in the latent class, they will return to the susceptible class, and are again susceptible to the zombie virus. This counter-virus will be applied to the model for latent infection in section 4.2.2. The differential equations for this model which S, L, Z and R satisfy are given by,

$$\frac{dS}{dt} = -\beta SZ + cL, \tag{4.3.2.1}$$

$$\frac{dL}{dt} = u\beta SZ - \rho L - cL, \tag{4.3.2.2}$$

$$\frac{dZ}{dt} = \rho L + \mu R - \alpha SZ, \tag{4.3.2.3}$$

$$\frac{dR}{dt} = q\beta SZ - \mu R. \tag{4.3.2.4}$$

In equation 4.3.2.2, the substitution for u given in equation 4.1.2.9 has been used. This model may be represented diagrammatically as shown in figure 27.

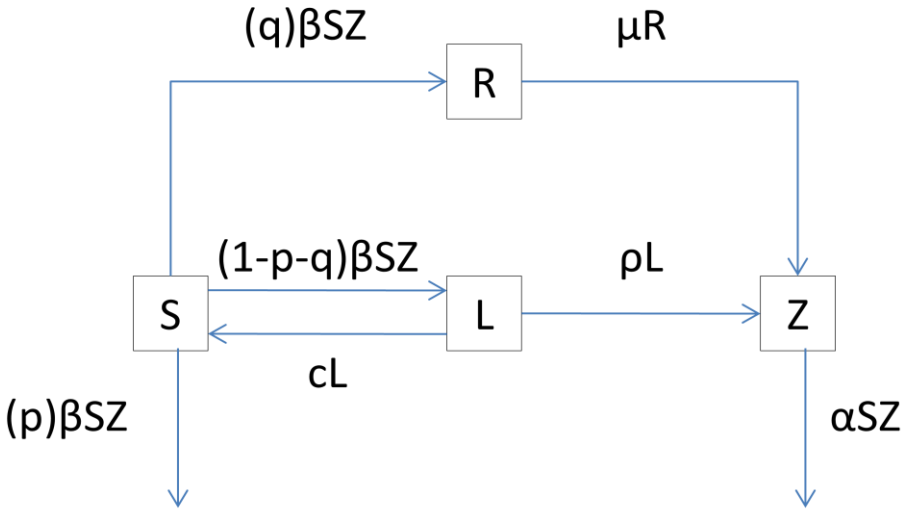

Figure 27: Flowchart for the Alternative SLZR model with a counter-virus.

The steady states of the differential equations for this model are found to be the same as those given in equations 4.2.1.5 and 4.2.1.6. Thus, there is no steady state for coexistence in this model, hence only

the susceptible or the zombie class may survive indefinitely. To determine the stability of the steady states for this model, the Jacobian matrix will be used and is given by,

$$J = \begin{bmatrix} -\beta Z & c & -\beta S & 0 \\ u\beta Z & -\rho - c & u\beta S & 0 \\ -\alpha Z & \rho & -\alpha S & \mu \\ q\beta Z & 0 & q\beta S & -\mu \end{bmatrix}, \tag{4.3.2.5}$$

which, when evaluated at the disease free steady state from equation 4.2.1.5 gives,

$$J(N,0,0,0) = \begin{bmatrix} 0 & c & -\beta N & 0 \\ 0 & -\rho - c & 0.85\beta N & 0 \\ 0 & \rho & -\alpha N & \mu \\ 0 & 0 & 0.1\beta N & -\mu \end{bmatrix}. \tag{4.3.2.6}$$

Next the equation for the eigenvalues is calculated as,

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda\left[\det\left(\begin{bmatrix} -\rho - c - \lambda & u\beta N & 0 \\ \rho & -\alpha N - \lambda & \mu \\ 0 & q\beta N & -\mu - \lambda \end{bmatrix}\right)\right],$$

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda[(-\rho - c - \lambda)[(-\alpha N - \lambda)(-\mu - \lambda) - q\beta\mu N] - (u\beta N)[\rho(-\mu - \lambda)]],$$

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda(-\lambda^3 - a_1\lambda^2 - a_2\lambda - a_3), \tag{4.3.2.7}$$

where,

$$a_1 = (c + \rho + \mu + \alpha N), \tag{4.3.2.8}$$

$$a_2 = \big((\rho + c + \mu)\alpha N + (\rho + c)\mu - (q\mu + u\rho)\beta N\big), \tag{4.3.2.9}$$

$$a_3 = \big((\rho + c)\mu\alpha N - (qc + (1 - p)\rho)\mu\beta N\big). \tag{4.3.2.10}$$

Thus for the disease free steady state to be stable, the inequality in equation 4.2.1.11 must be satisfied by equation 4.3.2.10. Therefore, the inequality,

$$\alpha > \frac{u\rho + qc}{c + \rho}\beta, \tag{4.3.2.11}$$

must be satisfied. This shows that the susceptible population must kill the zombies at a sufficiently fast rate in order to survive and prevent the Class 2 outbreak from escalating. Upon simulating this model, it will be shown that equation 4.2.1.12 is also satisfied.

Evaluating the Jacobian matrix at the apocalyptic steady state given in equation 4.2.1.6,

$$J(0,0,Z^*,0) = \begin{bmatrix} \beta Z^* & c & 0 & 0 \\ u\beta Z^* & -\rho - c & 0 & 0 \\ -\alpha Z^* & \rho & 0 & \mu \\ q\beta Z^* & 0 & 0 & -\mu \end{bmatrix}, \tag{4.3.2.12}$$

from which the equation for the eigenvalues may be determined. This is given as,

$$\det(J(0,0,Z^*,0)-\lambda I_4)=(-\lambda)\left[\det\left(\begin{bmatrix}-\beta Z^*-\lambda & c & 0\\ u\beta Z^* & -\rho-c-\lambda & 0\\ q\beta Z^* & 0 & -\mu-\lambda\end{bmatrix}\right)\right],$$

$$\det(J(0,0,Z^*,0)-\lambda I_4)=-\lambda[(-\mu-\lambda)[(-\beta Z^*-\lambda)(-\rho-c-\lambda)-uc\beta Z^*]],$$

$$\det(J(0,0,Z^*,0)-\lambda I_4)=-\lambda(-\lambda^3-a_1\lambda^2-a_2\lambda-a_3), \tag{4.3.2.13}$$

where the coefficients are given as,

$$a_1=(c+\rho+\mu+\beta Z^*), \tag{4.3.2.14}$$

$$a_2=(\mu\rho+c\mu+\mu\beta Z^*+\rho\beta Z^*+c\beta Z^*-uc\beta Z^*), \tag{4.3.2.15}$$

$$a_3=(\mu\rho\beta Z^*+(1-u)\mu c\beta Z^*). \tag{4.3.2.16}$$

Hence, the inequalities in equations 4.2.1.10, 4.2.1.11, and 4.2.1.12 are always satisfied when the system is physical. Thus, the apocalyptic steady state is always stable.

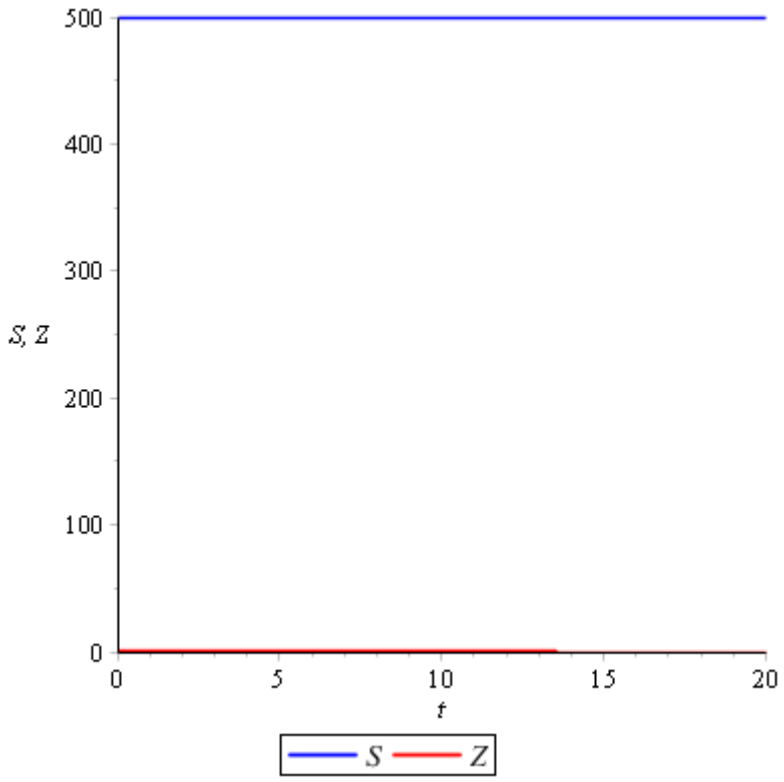

Figure 28: Simulation of the Alternative SLZR model with a counter-virus, with stable disease free steady state.

In figure 28, the simulation was based on the same assumptions for the parameters as in the simulation shown in figure 17 from section 4.1.2. Hence it was chosen that, $\beta = 0.005$,$\alpha = 0.007$, $\mu = \rho = 1$, $p = 0.05$, $q = 0.1$. It was also assumed in this simulation that it took, on average, 2 days

for an individual to be cured in the latent class, and hence $c = 0.5$. Thus, the inequality in equation 4.2.1.12 becomes, $44.5 > 0$, using equations 4.3.2.8, 4.3.2.9, and 4.3.2.10. Given that the inequality in equation 4.3.2.11 is also satisfied, the disease free steady state is stable. Hence, figure 28 shows that the zombie horde is eradicated in approximately 14 days, which is only 1 day faster than for the simulation of the Basic SZR model shown in figure 17. However, 1 day is still faster, thus suggesting that the cure is beneficial to the eradication of the zombie horde and prevents the outbreak from growing larger than a Class 1 outbreak.

Figure 29 shows a simulation of the alternative cure model where the apocalyptic steady state is reached. In order for this steady state to be reached, the assumptions made on the population were the same as for the simulation in figure 18 from section 4.1.2. Thus, the same values for the parameters where used, with the addition that $c = 0.5$ by the same assumptions made in the simulation in figure 28. Thus, the inequality in equation 4.3.2.11 is unsatisfied and this shows that the susceptible population is overwhelmed by the zombies after approximately 150 days. This shows a significant increase in the survival time, and a decrease in the number of zombies present at the steady state when compared to figures 15, 18, 20 and 23.However, this outbreak becomes a little more difficult to classify since there is approximately 50 zombies present at the steady state, yet total loss of life in the susceptible population. The number of zombies created over the course of the outbreak fits the classification for a Class 2 outbreak, but only fits the classification for a Class 1 outbreak at the steady state. The crucial point however, is that there are still zombies present in the population, and hence this could still result in a Class 4 outbreak if left unchecked by some external force.

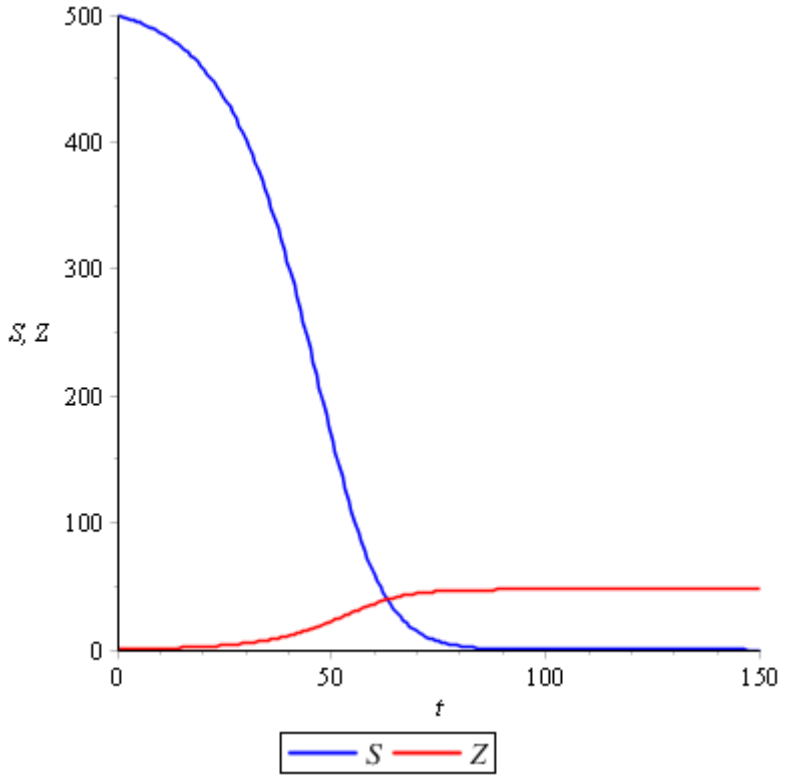

Figure 29: Simulation of the Alternative SLZR model with a cure, where the disease free steady state is unstable.

5 The Basic SYZ Model

In order to extend the previous models which have been analysed, a crucial change will now be made to the structure of the models. Since it has been the case in each scenario, throughout section 4, that the rate at which an individual leaves the latent and removed classes has been assumed to be the same, these two classes will now be condensed into a single class, Y. The rate at which an individual leaves the Y class will be given by the parameter ρ. Thus, instead of entering the removed class, the latent class, or leaving the population due to destruction of the brain, it will now be assumed that all susceptible individuals enter the Y class on losing an interaction with a zombie. Hence, the basic model for an outbreak of a zombie virus, that will be extended to include other population classes, is given by the differential equations,

$$\frac{dS}{dt} = -\beta SZ, \tag{5.1}$$

$$\frac{dY}{dt} = \beta SZ - \rho Y, \tag{5.2}$$

$$\frac{dZ}{dt} = \rho Y - \alpha SZ. \tag{5.3}$$

This model may be represented diagrammatically as shown in figure 30.

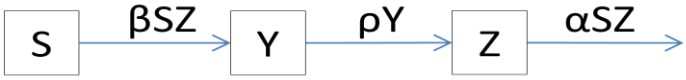

Figure 30: Flowchart representing the basic SYZ model for zombie infection.

The steady states of this model are given by,

$$(S^*, Y^*, Z^*) = (N, 0, 0), \tag{5.4}$$

$$(S^*, Y^*, Z^*) = (0, 0, Z^*). \tag{5.5}$$

Upon analysing the Jacobian matrix for this model, it is found that the apocalyptic steady state in equation 5.5 is always stable with eigenvalues,

$$\lambda_1 = 0, \lambda_2 = -\rho, \lambda_3 = -\beta Z^*,$$

where , λ_2, λ_3, are negative and real and $\lambda_1 = 0$.

Analysis of the Jacobian matrix evaluated at the disease free steady state in equation 5.4 gives,

$$\det(J(N,0,0) - \lambda I_3) = -\lambda\left[\det\left(\begin{bmatrix} -\alpha N - \lambda & \rho \\ \beta N & -\rho - \lambda \end{bmatrix}\right)\right],$$

$$\det(J(N,0,0) - \lambda I_3) = -\lambda(\lambda^2 + a_1\lambda + a_2), \tag{5.6}$$

with the coefficients a_1, a_2, given by,

$$a_1 = \rho + \alpha N, \tag{5.7}$$

$$a_2 = (\alpha - \beta)\rho N. \tag{5.8}$$

Since $a_2 > 0$, for the stability of the disease free steady state by equation 4.1.1.15, it can be deduced from equation 5.8 that,

$$\alpha > \beta. \tag{5.9}$$

Hence, it is found that for humanity to survive the zombie outbreak, and prevent the escalation of the outbreak, the susceptible population must kill zombies at a rate which is faster than the rate at which zombies infect susceptibles.

Figure 31 shows a simulation of this model with values of the parameters chosen such that the disease free steady state is stable. It was assumed that there are 500 susceptible individuals present in the population when $t = 0$, and just a single zombie. Hence the initial conditions were that, $S(0) = 500$, $Y(0) = 0$, and $Z(0) = 1$. It was chosen that $\rho = 1$, since it is assumed that it would take an individual one day, on average, to leave the Y class. Also it was assumed that a single zombie introduced into a population of 500 susceptible individuals would be able to make 2.5 contacts sufficient to transmit the infection per day. Hence, $\beta = 0.005$. Hence, it was chosen that $\alpha = 0.007$, giving the term $\alpha S(0)Z(0) = 3.5$, and that the inequality in equation 5.9 is satisfied. Hence resulting in the

destruction of 3.5 zombies per day under the initial conditions. This simulation finds that the zombie threat is eradicated in less than 15 days, with only a small number of the fatalities in the susceptible population, thus eradicating the zombie horde and only resulting in a Class 1 outbreak.

Figure 32 shows a simulation of this model where the disease free steady state is unstable, resulting in the apocalyptic steady state being reached. The initial conditions were retained from the simulation in figure 31, along with the values for parameters β and ρ. However, the value for parameter α was reduced so that $\alpha = 0.003$, since it is now assumed that a population of 500 susceptibles, with a single zombie present, would be able to kill 1.5 zombies on average per day. Thus, the inequality in equation 5.9 is no longer satisfied, resulting in the instability of the disease free steady state. Here, the susceptible population is overwhelmed by the zombie horde in less than 30 days, with the number of zombies at the steady state sufficient for a Class 2 outbreak.

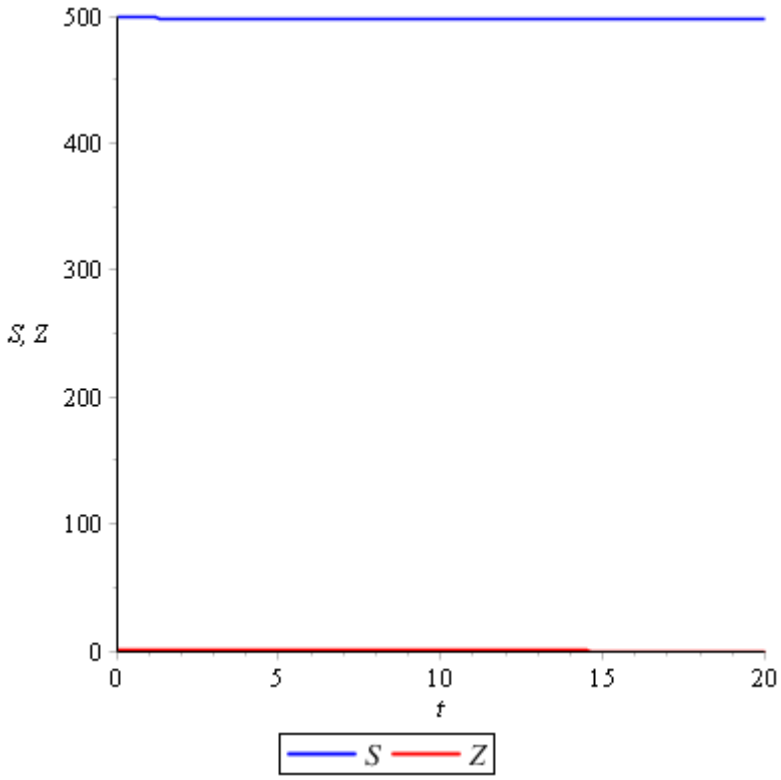

Figure 31: Simulation of the SYZ model where the disease free steady state is stable.

This model finds that the susceptible population can only survive the zombie apocalypse if they can kill zombies at a faster rate than the rate at which zombies infect susceptibles. However, there are several extensions which could be made to the model. These extensions, if beneficial to the susceptible population, should reduce the required value of α for the disease free steady state to be stable. However, there may be extensions which prove to be a hindrance, and increase the required value of α. Unless stated otherwise, the simulations throughout section 5 and its subsections will all

use the same initial conditions as the simulation shown in figure 31, and the same values for the parameters β and ρ.

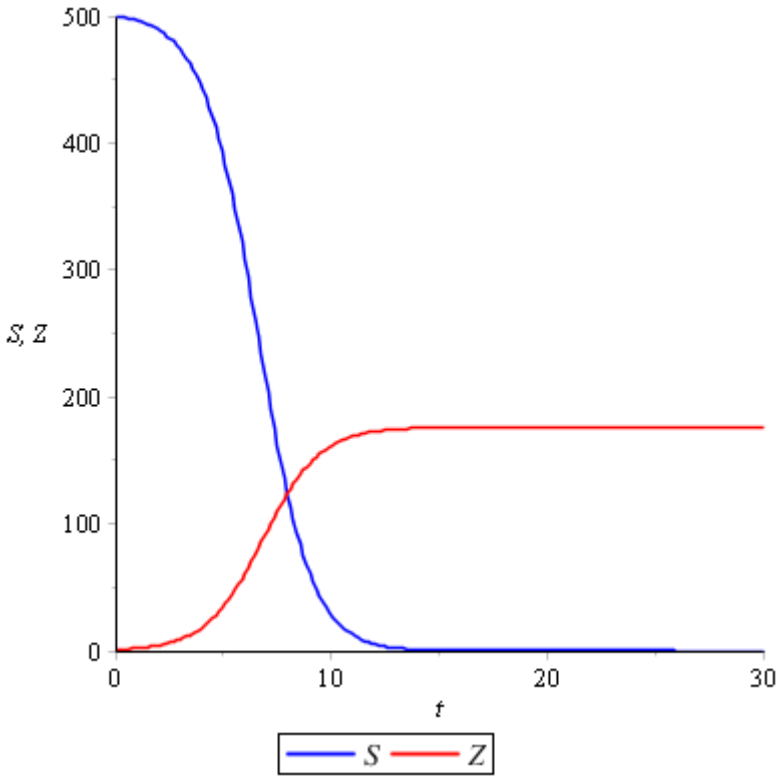

Figure 32: Simulation of the SYZ model where the disease free steady state is unstable.

5.1 The SBYZ Model

The new assumption introduced in this model is that once infected, there is a chance that an individual will attempt to kill zombies at an increased rate. The individual may do this as an attempt to save friends or family from the zombie horde, sacrificing themselves in the process (AMC, 2012-2013). To model this, a new class called the blaze class, B, will be introduced to the Basic SYZ model outlined in section 5 by equations 5.1, 5.2 and 5.3. It will be assumed that an individual enters the blaze class with a given probability, q, upon losing an interaction with a zombie, and that whilst in the blaze class, they kill zombies at a rate given by parameter η. However, since the individual has been infected, they will only have a limited amount of time spent in this class. Assume that the zombification process occurs at the same rate for an individual in the B class as it does for an individual in the Y class. Thus, individuals in the blaze class will exit the class at a rate given by parameter ρ. It will be assumed that all those in the blaze class take their own lives prior to becoming a zombie, and that they do so in a way which prevents them from reanimating. Due to the short

amount of time spent in the blaze class, and the nature of the actions of an individual in this class, there will not be a term included for a zombie attacking an individual who belongs to class B. The differential equations for this model are,

$$\frac{dS}{dt} = -\beta SZ, \tag{5.1.1}$$

$$\frac{dB}{dt} = q\beta SZ - \rho B, \tag{5.1.2}$$

$$\frac{dY}{dt} = (1-q)\beta SZ - \rho Y, \tag{5.1.3}$$

$$\frac{dZ}{dt} = \rho Y - \alpha SZ - \eta BZ. \tag{5.1.4}$$

The flowchart for this model is given in figure 33.

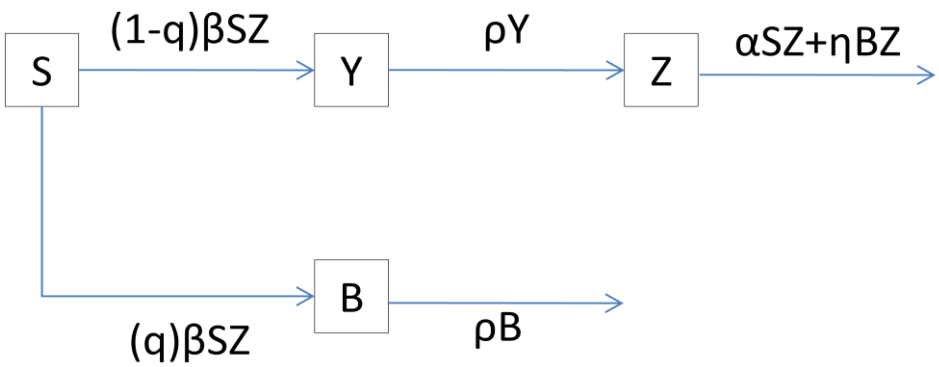

Figure 33: Flowchart for the SBYZ model of zombie infection.

The steady states of this model are given by,

$$(S^*, B^*, Z^*, Y^*) = (N, 0, 0, 0), \tag{5.1.5}$$

$$(S^*, B^*, Z^*, Y^*) = (0, 0, Z^*, 0). \tag{5.1.6}$$

Notice that $B^* = 0$, in both equation 5.1.5 and equation 5.1.6, since the individuals which enter B will still become deceased regardless of the presence of zombies due to the term, $-\rho B$. While this additional class has not resulted in a new steady state for the model, the value for α required for the disease free steady state to be stable may be reduced. To analyse the stability of the steady states, the Jacobian matrix will be used, and is given as,

$$J(S,B,Z,Y) = \begin{bmatrix} -\beta Z & 0 & -\beta S & 0 \\ q\beta Z & -\rho & q\beta S & 0 \\ -\alpha Z & -\eta Z & -\eta B - \alpha S & \rho \\ (1-q)\beta Z & 0 & (1-q)\beta S & -\rho \end{bmatrix}. \tag{5.1.7}$$

Calculating, $\det(J(0,0,Z^*,0) - \lambda I_4)$, gives the eigenvalues of the apocalyptic steady state as,

$$\lambda_1 = 0, \lambda_2 = -\rho, \lambda_3 = -\rho, \lambda_4 = -\beta Z^*,$$

which gives that the apocalyptic steady state is always stable since the eigenvalues have negative real parts.

Evaluating the Jacobian matrix at the disease free steady state gives,

$$J(N,0,0,0) = \begin{bmatrix} 0 & 0 & -\beta S & 0 \\ 0 & -\rho & q\beta S & 0 \\ 0 & 0 & -\alpha S & \rho \\ 0 & 0 & (1-q)\beta S & -\rho \end{bmatrix}, \tag{5.1.8}$$

which can then be used to calculate the eigenvalue equation,

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda \left[\det\left(\begin{bmatrix} -\rho-\lambda & -q\beta N & 0 \\ 0 & -\alpha N-\lambda & \rho \\ 0 & (1-q)\beta N & -\rho-\lambda \end{bmatrix} \right) \right],$$

$$\det(J(N,0,0,0) - \lambda I_4) = -\lambda(-\rho-\lambda)(\lambda^2 + a_1\lambda + a_2), \tag{5.1.9}$$

where,

$$a_1 = \alpha N + \rho, \tag{5.1.10}$$

$$a_2 = \alpha\rho N - (1-q)\beta\rho N. \tag{5.1.11}$$

For the stability of the disease free steady state, equations 4.1.1.14 and 4.1.1.15 must be satisfied. Thus, from equation 5.1.11 it is found that the inequality,

$$\alpha > (1-q)\beta, \tag{5.1.12}$$

must be satisfied for the disease free steady state to be stable.

The inequality in equation 5.1.12 shows that increasing the probability that an individual enters the blaze class, reduces the required rate for α in order for the disease free steady state to be stable. However, the analysis gives no indication of what will happen if the parameter η takes different values. This will be investigated in the simulations of this model.

The first simulation will assume that $\eta = 2\alpha$. This means that while an individual is in the blaze class, they kill zombies at twice the rate of an individual in the susceptible class. The assumptions based on the rate at which the susceptible population is able to destroy zombies are the same as in the simulation in figure 31 from section 5. Parameters where chosen such that, $\beta = 0.005$, $\alpha = 0.007$, $\eta = 0.014$, $\rho = 1$. So in this model, an individual will spend, on average, one day in the blaze class before they sacrifice themselves. It has been assumed that 20% of the susceptible population would enter the blaze class, hence $q = 0.2$. These parameters were sufficient for the stability of the disease free steady state, since equation 5.1.12 is satisfied under these assumptions. This simulation is shown in figure 34, and the outbreak never escalates beyond the classification for a Class 1 outbreak.

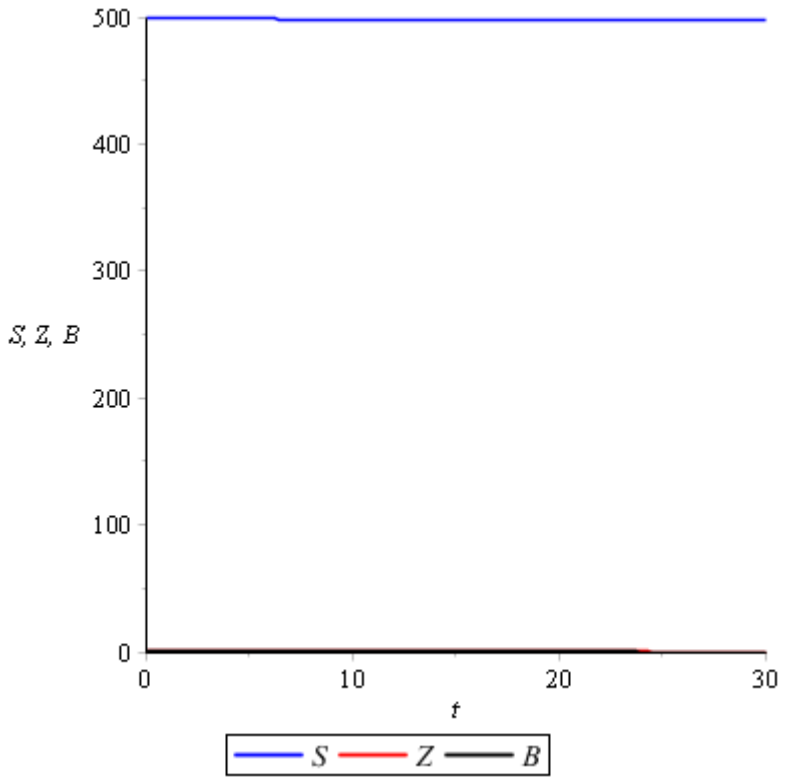

Figure 34: Simulation of the SBZY model with stable disease free steady state.

However, a simulation of this model with parameters chosen such that the disease free steady state is unstable is given in figure 35. The same values for the parameters were used as in the simulation in figure 34, with the exception that $\alpha = 0.003$, and $\eta = 0.006$, which results in equation 5.1.12 being unsatified. The value for parameter α was based on the assumptions made in the simulation of figure 32 from section 5. The result of the low rate for α is a Class 2 outbreak.

It is also found that regardless of the value chosen for parameter η, if α is not chosen such that the inequality in equation 5.1.12 is satisfied, then the apocalyptic steady state will still be reached. In figure 36, it was chosen that $\alpha = 0.003$, and $\eta = 300\alpha = 0.9$, and hence an individual in the blaze class destroys zombies at 300 times the rate that a susceptible does. However, this assumption may be unreasonable if spatial effects were to be considered. The timescale of the simulation is approximately five years, which is a long enough time scale for natural birth and death rates to be reasonably included. However, the capability with which a five year old could destroy a zombie is questionable at best, hence there will be no allowance for natural birth rates within the model.

The addition of the blaze class is a physically realistic assumption to make (AMC, 2012-2013). Comparing the simulations in figure 31 and 34 finds that the blaze model takes a longer period of time to eradicate the zombies than the basic model does. However, comparing figure 32 with figure 35 finds that the blaze model extends the survival of humanity by approximately 40 days. While not a

significant amount of time compared to the life expectancy of the average human, it is still allows more time for the susceptible population to find a better means of eradicating the zombie horde. However, this model still finds that unless certain conditions are met, an outbreak of zombies may still become a Class 2 outbreak, with the potential to escalate further.

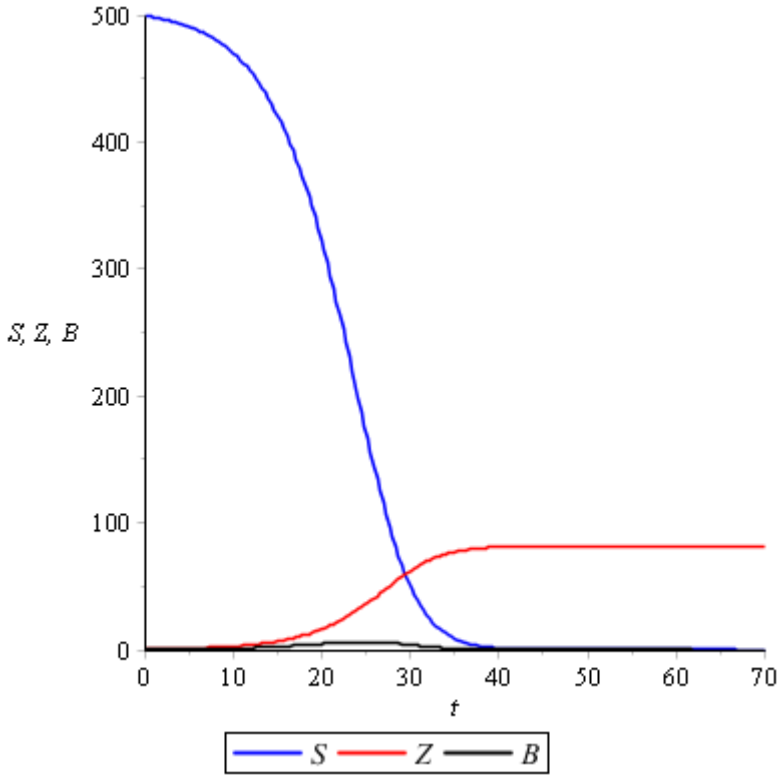

Figure 35: Simulation of the SBZY model where the disease free steady state is unstable.

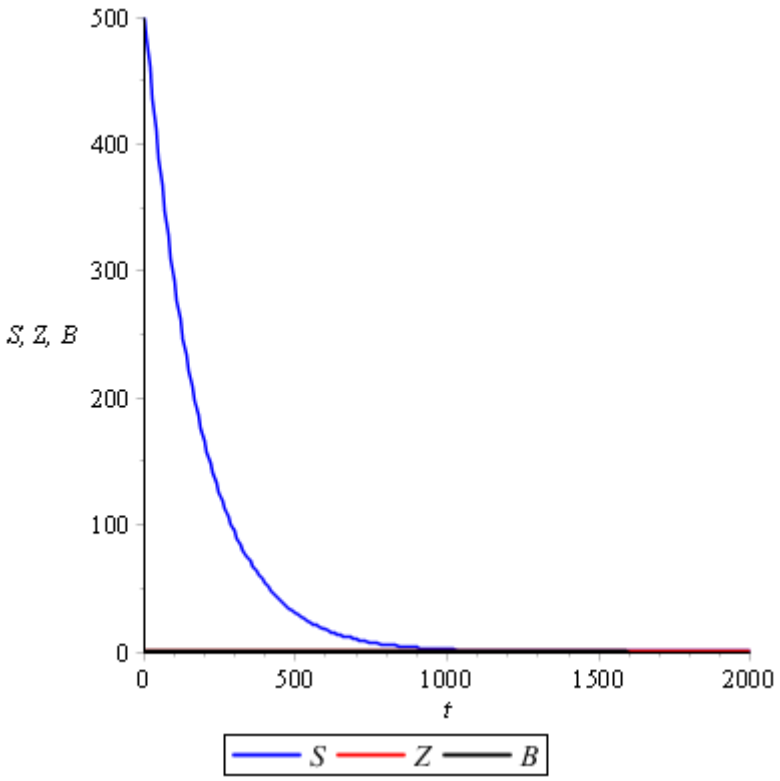

Figure 36: Simulation of the SBZY model where the disease free steady state is unstable.

5.2 The SFYZ Model

Suppose that the population has access to some item, such as a weapon, that allows the individual to destroy zombies at a rate, η. Assume that zombies are unable to use such a weapon. Now introduce a new class of fighters, F, which are susceptible individuals who have acquired one of these weapons. Let the rate at which a susceptible acquires a weapon be given by parameter σ. However, suppose there is only a finite number of weapons available, X, with $N > X$, where N is the population without infection. Whilst a fighter may kill zombies at a faster rate than a susceptible, it will be assumed that both fighters and susceptibles are infected at the same rate, given by parameter β. Upon infection, it will be assumed that a fighter surrenders their weapon, allowing a new susceptible to enter the fighter class. Hence, this model extends the Basic SYZ model outlined in section 5. The fighter class essentially behaves as the blaze class did in section 5.1, but an individual may be able to exist in the fighter class indefinitely, rather than the average time of one day spent in the blaze class. The differential equations for the fighter model are given by,

$$\frac{dS}{dt} = -\beta SZ - \sigma(X - F)S, \tag{5.2.1}$$

$$\frac{dF}{dt} = \sigma(X - F)S - \beta FZ, \tag{5.2.2}$$

$$\frac{dZ}{dt} = \rho Y - \alpha SZ - \eta FZ, \tag{5.2.3}$$

$$\frac{dY}{dt} = \beta Z(F + S) - \rho Y. \tag{5.2.4}$$

The SFYZ model is represented diagrammatically in figure 37.

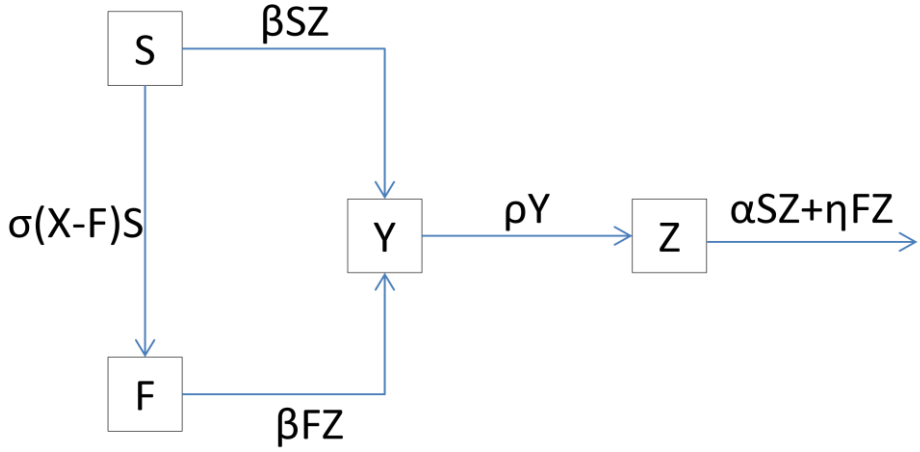

Figure 37: Flowchart for the SFYZ model of an outbreak of zombie infection.

The steady states of equations 5.2.1, 5.2.2., 5.2.3, and 5.2.4 are,

$$(S^*, F^*, Z^*, Y^*) = (U, X, 0, 0), \tag{5.2.5}$$

$$(S^*, F^*, Z^*, Y^*) = (0, 0, Z^*, 0), \tag{5.2.6}$$

$$(S^*, F^*, Z^*, Y^*) = (0, F^*, 0, 0), \tag{5.2.7}$$

where the substitution, $U = N - X$, has been made in equation 5.2.5.

The Jacobian matrix of equations 5.2.1, 5.2.2, 5.2.3, and 5.2.4 is given by,

$$J(S, F, Z, Y) = \begin{bmatrix} -\beta Z - \sigma(X - F) & \sigma S & -\beta S & 0 \\ \sigma(X - F) & -\sigma S - \beta Z & -\beta F & 0 \\ -\alpha Z & -\eta Z & -\eta F - \alpha S & \rho \\ \beta Z & \beta Z & \beta(S + F) & -\rho \end{bmatrix}. \tag{5.2.8}$$

The Jacobian can then be evaluated at the disease free steady state in order to determine conditions for its stability. Thus, evaluating the Jacobian matrix at the steady state given in equation 5.2.5 gives,

$$J(U, X, 0, 0) = \begin{bmatrix} 0 & \sigma(U) & -\beta(U) & 0 \\ 0 & -\sigma(U) & -\beta X & 0 \\ 0 & 0 & -\eta X - \alpha(U) & \rho \\ 0 & 0 & \beta N & -\rho \end{bmatrix}. \tag{5.2.9}$$

From equation 5.2.9, the eigenvalue equation is determined as,

$$\det(J(U, X, 0, 0) - \lambda I_4) = -\lambda \left[\det\left(\begin{bmatrix} -\sigma(U) - \lambda & -\beta X & 0 \\ 0 & -\eta X - \alpha(U) - \lambda & \rho \\ 0 & \beta N & -\rho - \lambda \end{bmatrix} \right) \right],$$

$$\det(J(U,X,0,0) - \lambda I_4) = -\lambda(-\lambda^3 - a_1\lambda^2 - a_2\lambda - a_3), \tag{5.2.10}$$

where the coefficients a_1, a_2, a_3, are given as,

$$a_1 = (\rho + \alpha U + \eta X), \tag{5.2.11}$$

$$a_2 = ([\sigma\alpha]U^2 + [\sigma\rho + \sigma\eta X + \alpha\rho]U + [\eta\rho X - \beta\rho N]), \tag{5.2.12}$$

$$a_3 = ([\sigma\alpha\rho]U^2 + [\sigma\eta\rho X - \sigma\beta\rho N]U). \tag{5.2.13}$$

For this steady state to be stable, the inequality in equation 4.2.1.11 must be satisfied. Hence, from equations 4.2.1.11 and 5.2.13 it is found that,

$$a_3 = ([\sigma\alpha\rho]U^2 + [\sigma\eta\rho X - \sigma\beta\rho N]U) > 0,$$

$$\alpha(N - X) + \eta X > \beta N, \tag{5.2.14}$$

must be satisfied for the disease free steady state to be stable.

However, there are two possible scenarios which will be considered. First, if $\eta = M\alpha$, where M is a positive, real number, then equation 5.2.14 becomes,

$$\alpha > \frac{\beta N}{N + [M-1]X}\ , \tag{5.2.15}$$

which gives the condition for stability under this assumption.

Secondly it may be that, $\eta = \alpha$, thus meaning that the weapons provide no increase in the rate at which an individual may kill a zombie. Thus, equation 5.2.14 becomes,

$$\alpha > \beta, \tag{5.2.16}$$

which is sufficient for stability of the disease free steady state, and is identical to equation 5.9 for the stability of the disease free steady state of the SYZ model.

It is also required that equation 4.2.1.12 is satisfied for stability in both these cases. However, this will be shown when numerical values are determined for the simulation of the model.

Evaluating the Jacobian matrix from equation 5.2.8 at the apocalyptic steady state given in equation 5.2.6, finds that the steady state will always be stable, with eigenvalues,

$$\lambda_1 = -\beta Z^*, \lambda_2 = -\rho, \lambda_3 = -0, \lambda_4 = -\sigma X - \beta Z^*,$$

which all have negative real part.

The stability of the steady state in equation 5.2.7 can also be determined. The eigenvalue equation found, when evaluating the Jacobian matrix from equation 5.2.8 at the steady state is,

$$\det(J(0,F^*,0,0) - \lambda I_4) = -\lambda(\lambda^2 + (\rho + \eta F^*)\lambda + (\eta - \beta)\rho F^*),$$

which is stable provided, $\eta > \beta$, since the inequality in equation 4.1.1.15 must be satisfied, where $a_2 = (\eta - \beta)\rho F^*$. Simulations of this model show that unless $X > N$ at the steady state, then the

disease free steady state will be reached. Thus, this steady state is will not be reached if the disease free steady state is stable. However, none of the simulations conducted found that this steady state was reached.

Figure 38 shows a simulation where it has been assumed that $\eta = \alpha$. Hence, the inequality in equation 5.2.16 must be satisfied in order for humanity to survive. By the same assumptions as in figure 31 from section 5, it was chosen that $\alpha = \eta = 0.007, \beta = 0.005, \rho = 1$. The rate σ was chosen based on the assumption that a population consisting of 500 susceptible individuals would find 10 weapons per day, and it has been assumed that there are only 50 weapons available. Hence, $\sigma = 0.0004$ and $X = 50$. With these values chosen for the parameters, the inequality in equation 4.2.1.12 becomes, $7.97 > 0$, and the inequalities in equations 4.2.1.10 and 4.2.1.11 are satisfied, which is sufficient for the stability of the disease free steady state, and prevents the outbreak escalating above Class 1.

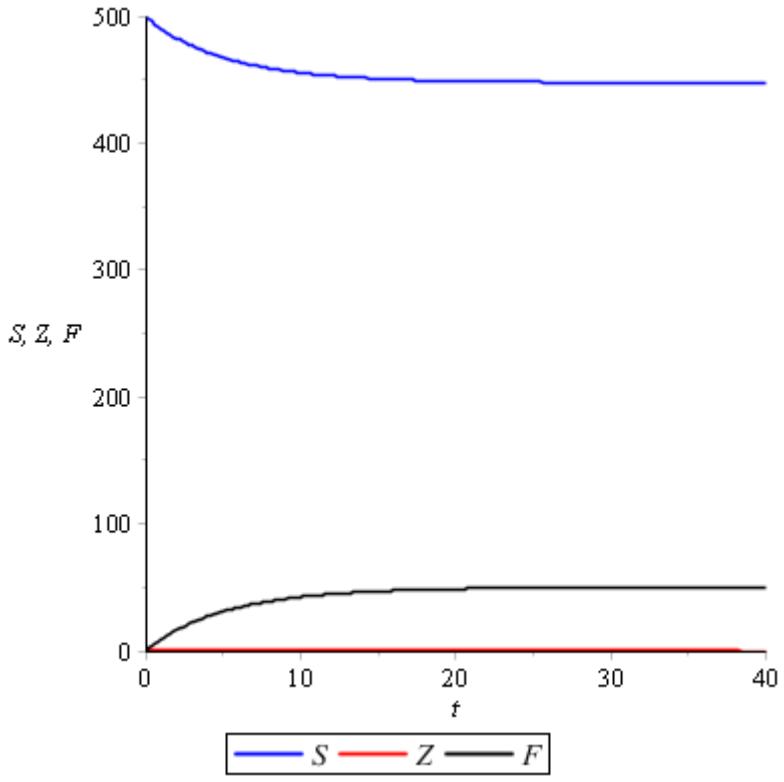

Figure 38: Simulation of the SFYZ model with $\alpha = \eta$, showing stability of the disease free steady state.

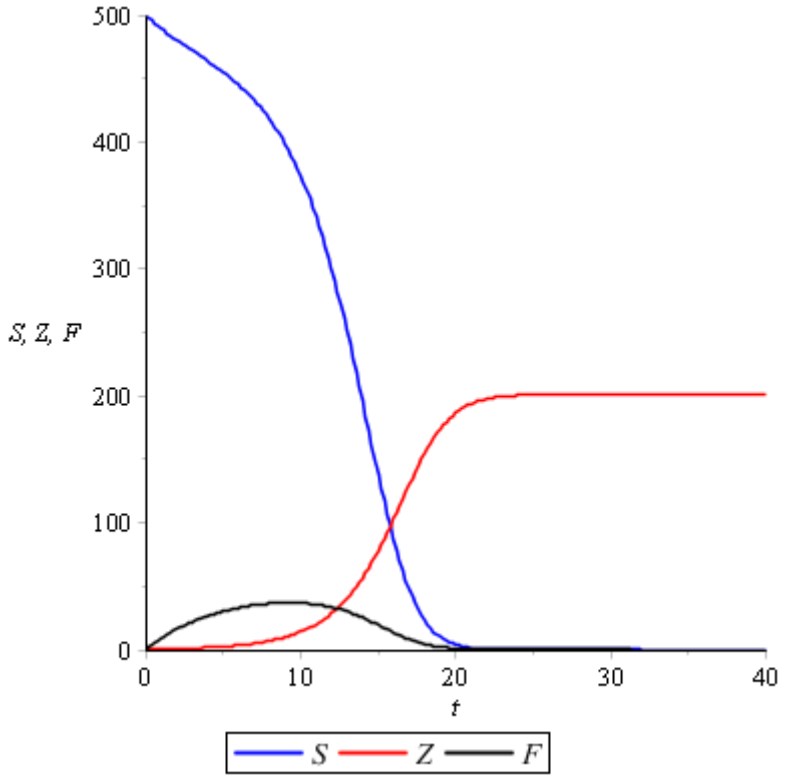

Figure 39: Simulation of the SFYZ model with $\alpha = \eta$, where the disease free steady state is unstable.

In figure 39, the same values for the parameters where used as in the simulation shown in figure 38, with the exception that $\alpha = \eta = 0.003$. This reduced rate for α is the result of making the same assumptions regarding the rate at which the susceptible population kills zombies as in the simulation shown in figure 32 from section 5. This was sufficient for the inequality in equation 4.2.1.11 to be unsatisfied, hence the disease free steady state became unstable. Whilst the number of fighters has peaked by time $t = 15$, it never quite reaches the limit, and the zombie population has been able to grow too large and overwhelms the human population. This results in the potential for the outbreak to escalate beyond a Class 2, and become a Class 3 or Class 4.

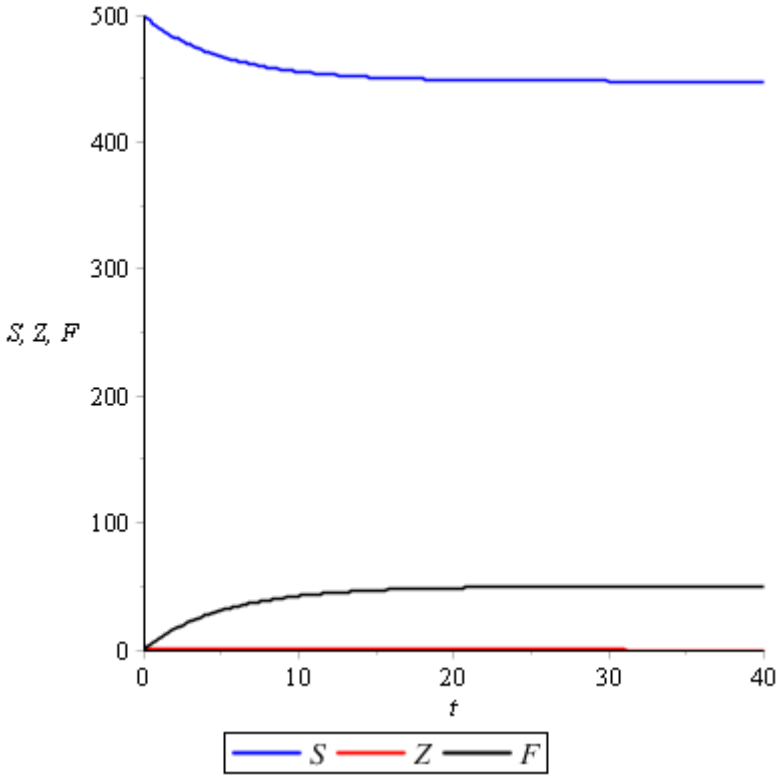

Figure 40: Simulation of the SFYZ model with $\eta = 2\alpha = 0.014$, and the disease free steady state is stable.

Suppose now that the weapons are beneficial in killing zombies, hence $\eta > \alpha$, so $\eta = M\alpha$ will be considered. Assuming $M = 2$, and then using the same values for parameters β, ρ, X and σ as in the simulation in figure 38, the disease free steady state is stable provided $\alpha > 0.0045$, by equation 5.2.15, and the inequality in equation 4.2.1.12 is satisfied. Thus, by the same assumptions on α as in the simulation in figure 31 from section 5, $\alpha = 0.007$, and hence $\eta = 0.014$. Since equation 4.2.1.12 becomes, $10.54 > 0$, the disease free steady state is stable. A simulation of this is shown in figure 40. This simulation finds that increasing the rate for η in such a manner reduces the timescale of the outbreak by approximately 8 days, and is able to prevent the outbreak growing larger than that of a Class 1 outbreak.

However, the beneficial weapons do not guarantee the survival of humanity as shown in figure 41. The simulation in figure 41 uses the same values for parameters as the simulation in figure 39, above, with the exception that $\alpha = 0.003$, and hence $\eta = 0.006$, by the same assumptions as made in the simulation in figure 32 from section 5. This shows that even with the fighter class being beneficial, humanity will still fall if they cannot kill the zombies at a sufficiently high rate. Although, it should be noted that whilst the zombies overwhelm the population, humanity survived for a longer time period than in the simulation in figure 39, where the weapons provided no benefit. However, such an

outbreak could still escalate from a Class 2 to a Class 4 global phenomena, if the outbreak continued unchecked once the steady state of this model has been reached.

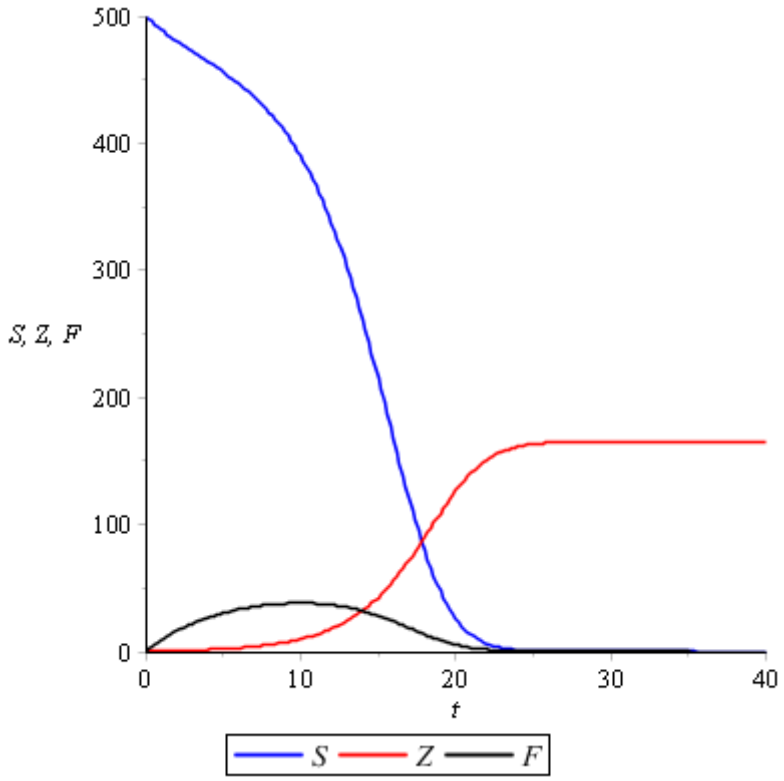

Figure 41: Simulation of the SFYZ with $\eta = 2\alpha = 0.006$, and the disease free steady state is unstable.

Comparing figure 41 with figure 35 from section 5.1, figure 41 shows a survival time of approximately 35 days, while figure 35 has a survival time of approximately 65 days. Thus, it appears that the blaze model is actually more beneficial to prolonging the survival of humanity than the fighter class is. In the blaze model, no individuals in the blaze class become a zombie, thus the total number of zombies expected in this model is reduced compared to the fighter model, since all fighters will still become zombies. The reduced number of zombies in the blaze model means that the number of new infections per day will be less than that in the fighter model, thus allowing more time for the survival of humanity. However, some changes could be made to the fighter model which would see it become more beneficial than the blaze model. Examples of these changes could be to reduce the rate at which a fighter may be infected by a zombie, or to increase the limit on the size of the fighter population.

5.3 The SZEY Model

This model will consider the consequences of a variant form of zombie. These variant zombies will be assumed to have capabilities and traits which are beyond that of the basic zombie (Valve, 2008; Valve, 2010). Thus, the implication is that the variations are beneficial to the individual zombie,

hence it will be referred to as an enhanced zombie and a new class, E, will be introduced into the model. Suppose that the enhanced zombies are harder to kill and are able to pass on the infection at a greater rate than the standard zombie. Two new parameters, γ and ϕ, will be introduced into the model, where $\gamma = M\alpha$ is the rate at which susceptibles destroy the enhanced zombies, and $\phi = D\beta$ is the rate at which the enhanced zombies infect susceptibles. Thus, the terms ϕSE and γSE are introduced, where ϕSE gives the number of new infections from enhanced zombies and γSE gives the number of enhanced zombies destroyed by susceptibles per unit time. It will be assumed that an individual becomes an enhanced zombie on leaving the Y class, and has a probability q of doing so. Finally, it will also be assumed that enhanced zombies do not attack other zombies. The differential equations used in the model are,

$$\frac{dS}{dt} = -\beta SZ - \phi SE, \tag{5.3.1}$$

$$\frac{dY}{dt} = \beta SZ + \phi SE - \rho Y, \tag{5.3.2}$$

$$\frac{dZ}{dt} = (1-q)\rho Y - \alpha SZ, \tag{5.3.3}$$

$$\frac{dE}{dt} = q\rho Y - \gamma SE. \tag{5.3.4}$$

This model is represented diagrammatically in figure 42.

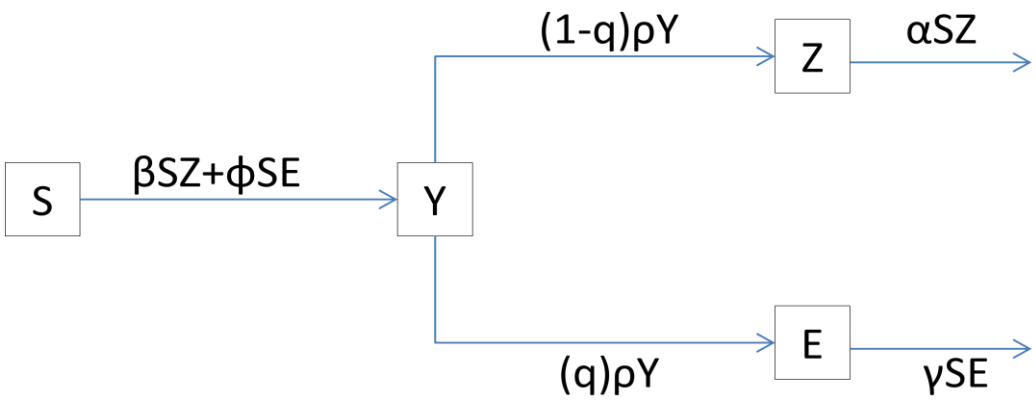

Figure 42: Flowchart of the SZEY model with the enhanced zombie population.

The steady states of the differential equations in equations 5.3.1, 5.3.2, 5.3.3, and 5.3.4 are given as,

$$(S^*, Z^*, E^*, Y^*) = (N, 0, 0, 0), \tag{5.3.5}$$

$$(S^*, Z^*, E^*, Y^*) = (0, Z^*, E^*, 0). \tag{5.3.6}$$

The Jacobian matrix for the differential equations in equations 5.3.1, 5.3.2, 5.3.3, and 5.3.4 is given by,

$$J(S,Z,E,Y)=\begin{bmatrix}-\beta Z-\phi E & -\beta S & -\phi S & 0\\ -\alpha Z & -\alpha S & 0 & (1-q)\rho\\ -\gamma E & 0 & -\gamma S & q\rho\\ \beta Z+\phi E & \beta S & \phi S & -\rho\end{bmatrix}, \tag{5.3.7}$$

which can then be analysed to determine the stability of the steady states.

Calculating, $\det(J(0,Z^*,E^*,0)-\lambda I_4)$, gives that the eigenvalues for the apocalyptic steady state in equation 5.3.6 are,

$$\lambda_1=0,\lambda_2=0,\lambda_3=-\rho,\lambda_4=-\beta Z^*-\phi E^*,$$

which is sufficient for the apocalyptic steady state to be stable, since all the eigenvalues have a negative real part.

Evaluating the Jacobian matrix from equation 5.3.7 at the disease free steady state given in equation 5.3.5 gives,

$$J(N,0,0,0)=\begin{bmatrix}0 & -\beta N & -\phi N & 0\\ 0 & -\alpha N & 0 & (1-q)\rho\\ 0 & 0 & -\gamma N & q\rho\\ 0 & \beta N & \phi N & -\rho\end{bmatrix}, \tag{5.3.6}$$

which is then use to calculate the eigenvalue equation. The eigenvalue equation is given as,

$$\det(J(N,0,0,0)-\lambda I_4)=-\lambda\left[\det\left(\begin{bmatrix}-\alpha N-\lambda & 0 & (1-q)\rho\\ 0 & -\gamma N-\lambda & q\rho\\ \beta N & \phi N & -\rho-\lambda\end{bmatrix}\right)\right],$$

$$\det(J(N,0,0,0)-\lambda I_4)=-\lambda(-\lambda^3-a_1\lambda^2-a_2\lambda-a_3), \tag{5.3.7}$$

where the coefficients a_1, a_2, and a_3, are,

$$a_1=(\alpha+\gamma)N+\rho, \tag{5.3.8}$$

$$a_2=\alpha\gamma N^2+\alpha\rho N+\gamma\rho N+(q-1)\beta\rho N-q\phi\rho N, \tag{5.3.9}$$

$$a_3=(\gamma-q\phi)\alpha\rho N^2+(q-1)\beta\gamma\rho N^2. \tag{5.3.10}$$

Given the initial assumptions of this model, the substitutions, $\gamma=M\alpha$ and $\phi=D\beta$, will be used. For stability of the disease free steady state, $a_3>0$, as given in equation 4.2.1.11, hence equation 5.3.10 gives that,

$$\alpha>\frac{M-qM+qD}{M}\beta\,, \tag{5.3.11}$$

must be satisfied, along with equation 4.2.1.12. This stability may be confirmed by simulating the model with various values chosen for the parameters.

In figure 43, the parameters were chosen such that the disease free steady state was stable. It was assumed that enhanced zombies would infect susceptibles at twice the rate of standard zombies, and that susceptibles could only kill enhanced zombies at half the rate that they killed standard zombies at.

Hence, $M = 0.5$, $D = 2$ were chosen. It was also assumed that there was a probability of 0.1 that an individual became an enhanced zombie, thus $q = 0.1$. As with the previous models, it was chosen that $\beta = 0.005$. Thus, in order to satisfy the inequality in equation 5.3.11, it was chosen that $\alpha = 0.007$, based on the same assumptions as for the simulation in figure 31 from section 5. The inequality in equation 4.2.1.12 becomes, $53.47 > 0$, which is satisfied, hence the disease free steady state is stable. Thus, even in the presence of the enhanced zombie class, a Class 1 outbreak may still be prevented from spreading and becoming a higher classification outbreak.

Figure 44 shows a simulation of the model where the disease free steady state is unstable. All previous values for parameters were retained from the simulation in figure 43, except it was now chosen that $\alpha = 0.003$, by the same assumptions made in the simulation shown in figure 32 from section 5. Thus, the inequality in equation 5.3.11 is no longer satisfied, which is sufficient for the disease free steady state to be unstable. This shows that it takes less than 35 days for the zombies to overwhelm humanity, and implies that without some external force or countermeasure, that the outbreak could escalate to a higher classification than a Class 2.

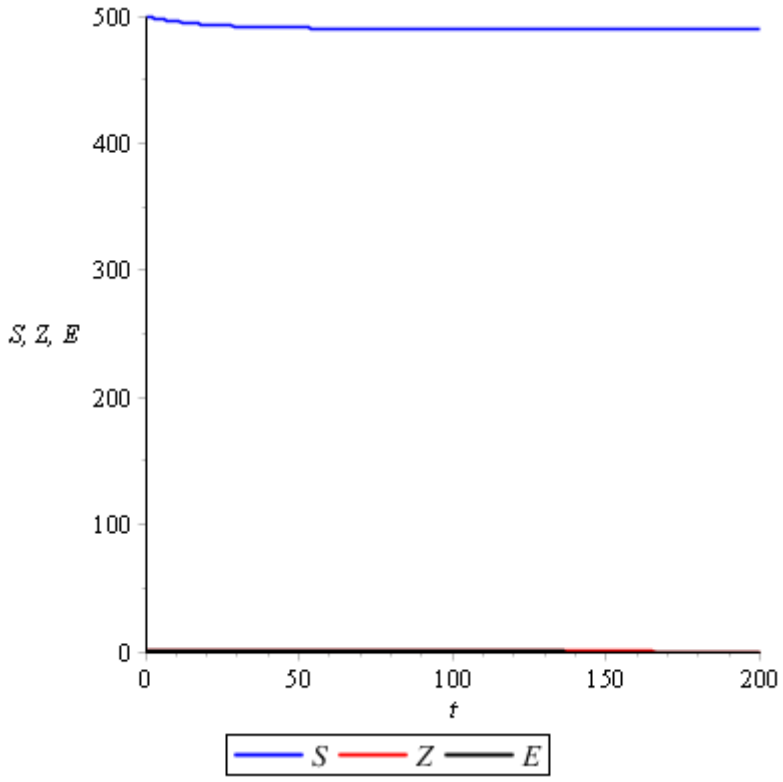

Figure 43: Simulation of the SZEY model with stable disease free steady state.

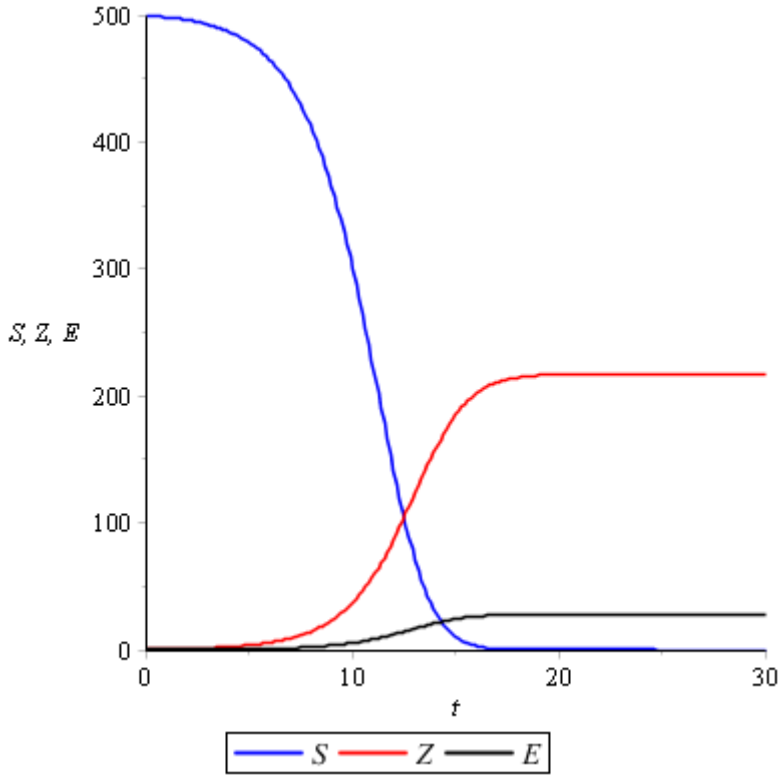

Figure 44: Simulation of the SZEY model with unstable disease free steady state.

These simulations have shown that, compared to the previous models seen throughout section 5, the presence of an enhanced class of zombie is detrimental to the survival of humanity. Comparing figure 43 with figure 31 in section 5, the timescale for the eradication of the zombies has been greatly increased. In the enhanced zombie model, it takes approximately 170 days for the zombie horde to be eradicated, and the susceptible population suffers greater losses than it did in the simulation of the Basic SYZ model in figure 31. Comparing figure 44 with figure 32 in section 5, the survival time of the susceptible population where the disease free steady state is unstable is also reduced, but not by a significant amount. The simulation in figure 32 finds that the population survives approximately 26 days, whilst only surviving for 25 days in the simulation of the enhanced model in figure 44. Thus, the presence of an enhanced zombie class makes countering the zombie horde more difficult, and hence could lead to a Class 2 outbreak or higher in a shorter time period than all previous models which have been considered throughout section 5.

5.4 The Anarchic SYZ Model

Suppose now that society has collapsed, and law and order have ceased (Bohemia Interactive, 2013; AMC, 2012-2013). A susceptible may fight another susceptible for a desired resource, or out of fear. It will be assumed that a susceptible kills another susceptible at a rate given by parameter δ, but that they also destroy the brain, preventing zombification of the fallen combatant. However, assume that they would only do so as long as the zombie population remains, meaning that in the absence of

zombies, there would be no combat between susceptibles. Thus a new term is introduced to the equations. The differential equations for this model are given by,

$$\frac{dS}{dt} = -\beta SZ - \frac{\delta S^2 Z}{Z+1}, \tag{5.4.1}$$

$$\frac{dY}{dt} = \beta SZ - \rho Y, \tag{5.4.2}$$

$$\frac{dZ}{dt} = \rho Y - \alpha SZ. \tag{5.4.3}$$

The term was chosen to be $\frac{\delta S^2 Z}{Z+1}$, so that in the absence of zombies, there is no division by zero. Likewise, susceptibles will only fight one another whilst the zombies are present due to the inclusion of a Z term in the numerator. The S^2 term in the numerator gives the interactions between one susceptible and another. Whilst parameter δ was chosen as the rate at which such an interaction occurs. This is represented diagrammatically in figure 45.

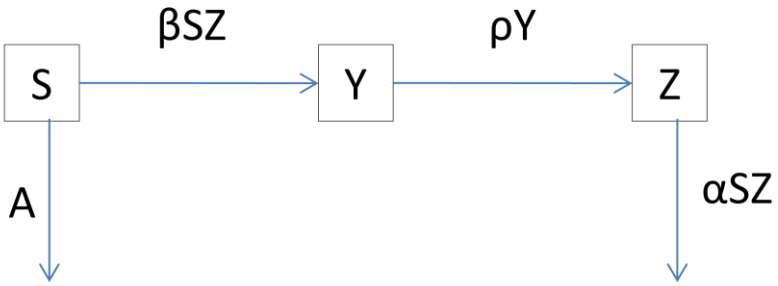

Figure 45: Flowchart for the Anarchic SYZ model, where $A = \frac{\delta S^2 Z}{Z+1}$.

The steady states for equations 5.4.1, 5.4.2, and 5.4.3 are equivalent to those from equations 5.4. and 5.5, where equation 5.4 gives the disease free steady state and equation 5.5 gives the apocalyptic steady state. The Jacobian matrix for equations 5.4.1, 5.4.2 and 5.4.3 is given as,

$$J = \begin{bmatrix} M & K & 0 \\ -\alpha Z & -\alpha S & \rho \\ \beta Z & \beta S & -\rho \end{bmatrix}, \tag{5.4.4}$$

where the substitutions,

$$M = -\beta Z - \frac{2\delta SZ}{Z+1}, K = -\beta S - \frac{\delta S^2}{Z+1} + \frac{\delta S^2 Z}{(Z+1)^2},$$

have been made.

Evaluating the Jacobian matrix from equation 5.4.4 at the apocalyptic steady state from equation 5.5, and determining the eigenvalues, gives that the apocalyptic steady state is always stable since the eigenvalues all have negative real part. The eigenvalues are given as,

$$\lambda_1 = 0, \lambda_2 = -\rho, \lambda_3 = -\beta Z^*.$$

Next evaluating the Jacobian matrix from equation 5.4.4 at the disease free steady state from equation 5.4 gives,

$$J(N,0,0) = \begin{bmatrix} 0 & -\delta N^2 - \beta N & 0 \\ 0 & -\alpha N & \rho \\ 0 & \beta N & -\rho \end{bmatrix}, \tag{5.4.5}$$

which can then be used to determine the eigenvalues for the steady state. Hence, the eigenvalue equation is given by,

$$\det(J(N,0,0) - \lambda I_3) = -\lambda\left[\det\left(\begin{bmatrix} -\alpha N - \lambda & \rho \\ \beta N & -\rho - \lambda \end{bmatrix}\right)\right],$$

$$\det(J(N,0,0) - \lambda I_3) = -\lambda(\lambda^2 + a_1\lambda + a_2), \tag{5.4.6}$$

which is identical to equation 5.6 with coefficients given in equations 5.7 and 5.8. Thus, the disease free steady state is stable, provided $\alpha > \beta$, as in equation 5.9. Hence the analysis suggests that the rate δ has no effect on the stability of this model. However, it should be noted that in practice, simulations show that increasing δ can cause the apocalyptic steady state to be reached.

Figure 46 shows the survival of the susceptible population, with values for the parameters chosen such that the disease free steady state is stable. The values chosen were, $\beta = 0.005, \alpha = 0.007, \rho = 1$, by the same assumptions made in the simulation shown in figure 31. It was also assumed that a population of 500 susceptible individuals, with just a single zombie present, would cause 10 susceptible individuals to be murdered as a result of the anarchic state. Hence parameter δ was chosen such that $\delta = 0.0008$. Thus, equation 5.9 is satisfied, resulting in the disease free steady state being stable. Whilst this results in the susceptible population surviving the zombie outbreak, it can be shown that an increased δ rate can cause the susceptible population to die out. It should be noted that this is only a Class 1 outbreak, despite the large number of susceptible fatalities, since the majority of the fatalities were caused by other susceptibles and not a direct result of zombie attacks.

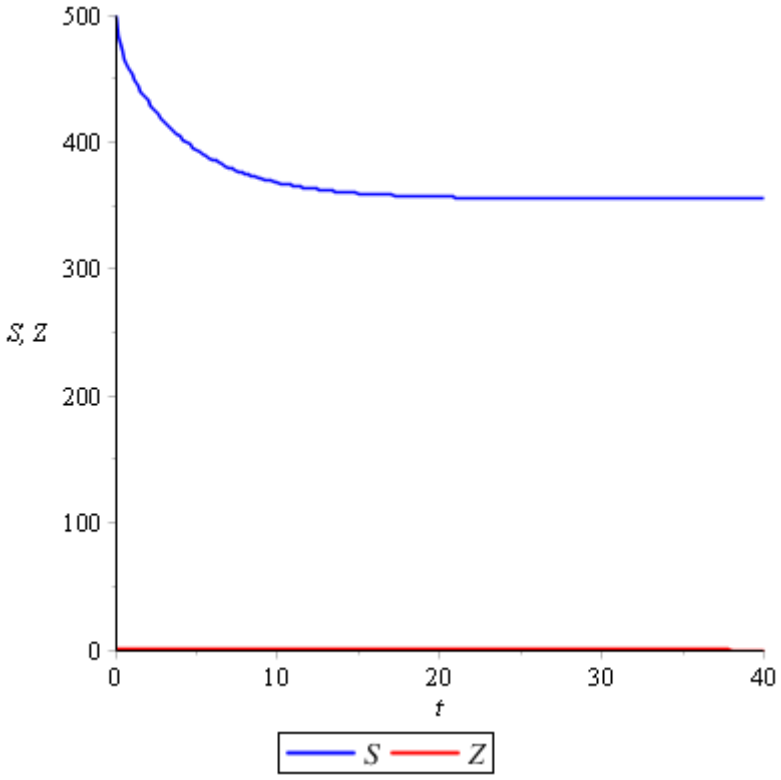

Figure 46: Simulation of the Anarchic SYZ model where the disease free steady state is stable.

Now assume that a population of 500 susceptible individuals, with a single zombie present, would result in 300 murders per unit time. This would mean that there are 300 susceptible fatalities as a result of the anarchic term. Hence, for the simulation in figure 47, it was chosen that $\delta = 0.01$, while all other parameter values and assumptions are the same as for the simulation in figure 46. Given that the initial condition, $S(0) = 500$, means that there are only five hundred susceptibles in the population, such a high rate is unreasonable. Whilst equation 5.9 is still satisfied, it shows that the parameter δ can affect the outcome of the simulation. Although, the simulation in figure 47 is considered to be an unlikely scenario, due to the high rate given to parameter δ. However, this outbreak never escalates above a Class 1 outbreak. This suggests that if there are other external populations which wish to prevent the spread of the zombie virus, finding a means of culling the susceptible population may be the best way of preventing the spread of the disease, and the escalation in classification of the outbreak.

Finally, a simulation where the conditions for the stability of the disease free steady state are not satisfied will be considered, and is shown in figure 48. Thus, equation 5.9 is not satisfied by the values for the parameters chosen. The parameters in this simulation took the same values as those for the simulation in figure 46, with the exception that $\alpha = 0.003$, were α was chosen based on the same assumptions as for the simulation shown in figure 32. This shows that the susceptible population are not killing the zombies at a sufficiently fast rate, and hence the apocalyptic steady state is reached.

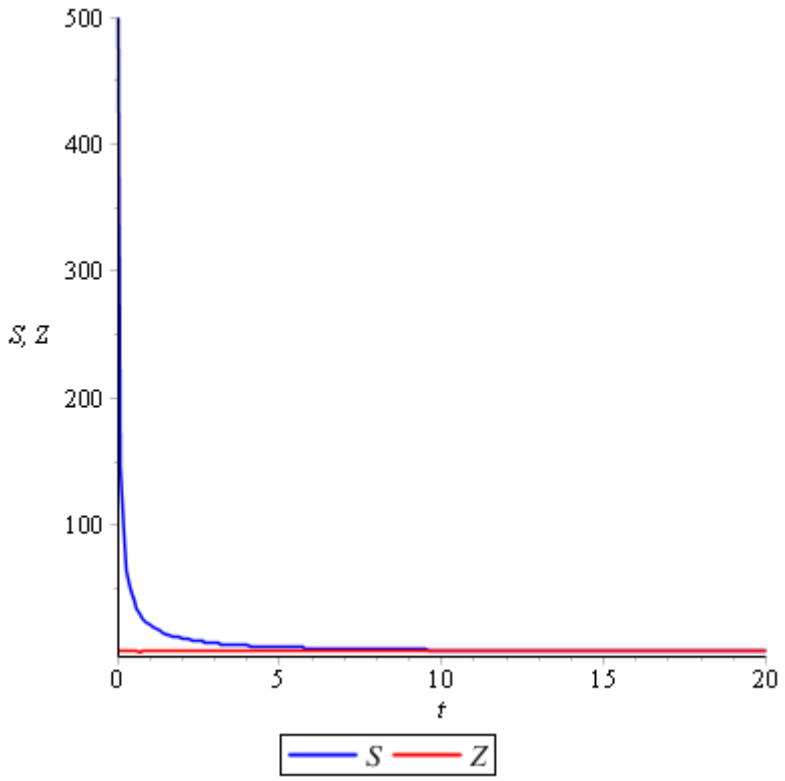

Figure 47: Simulation of the Anarchic SYZ model, where δ is so large that the susceptible population dies out.

The simulations of the Anarchic SYZ model confirm that an anarchic population is less able to confront a zombie horde, than an assumed unified population as seen in the Basic SYZ model in section 5.Comparing figure 31 and figure 46 finds that the susceptible population is significantly decreased at the disease free steady state in the anarchic model. However, this was to be expected due to the increased rate at which individuals leave the susceptible population in the anarchic model compared to the basic model. However, comparing figure 32 with figure 48 finds that the survival time of the susceptible population is increased by approximately ten days in the anarchic model. Also, the number of zombies present at the steady state in the anarchic model is decreased. The assumption that susceptible individual killing another susceptible prevents the zombification of the individual who is defeated, results in far few zombies being present at the disease free steady state than there are in figure 32 of the Basic SYZ model. This would suggest that less effort would be required from an external force to eradicate the zombie horde before the outbreak escalates. So whilst the anarchic model is detrimental to the population being modelled, it may prove beneficial to an external population and allow a Class 2 outbreak to be stopped by simpler means.

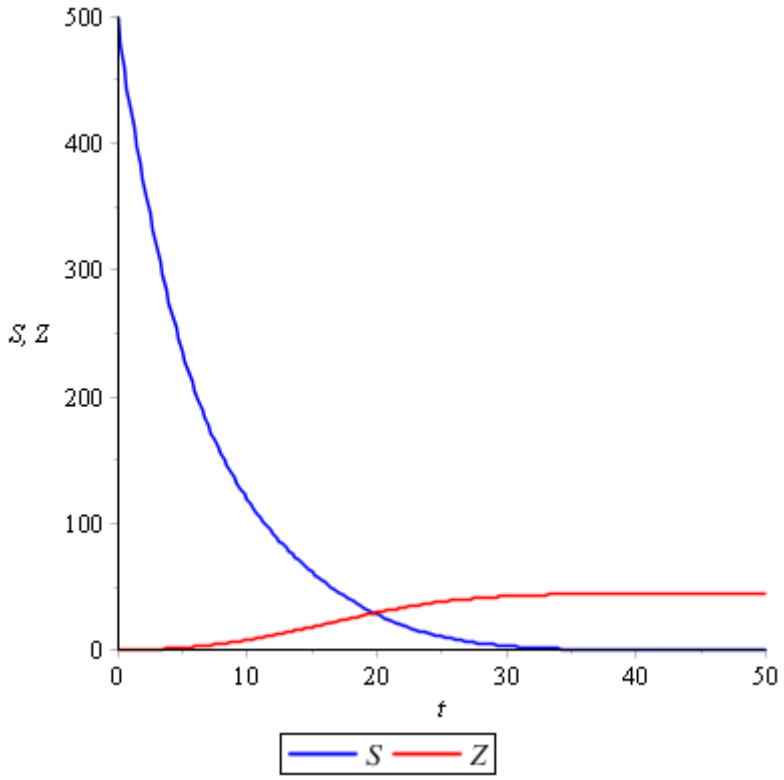

Figure 48: Simulation of the Anarchic SYZ model where the disease free steady state is unstable.

6. Discussion

The basic models shown in sections 4.1.1, 4.2.1 and 4.3.1 all make the assumption that a destroyed zombie may reanimate an infinite number of times. Hence it is not surprising that these models present no means of eliminating the zombie threat. However, since zombies posses no regenerative capabilities, as stated in section 2, this is assumed to be a fallacy. The models presented in sections 4.1.2, 4.2.2 and 4.2.3 sought to rectify this issue, and in turn found that humanity may prevail, and prevent a potential Class 2 outbreak, provided that the susceptible population is able to kill more zombies per day, than the zombies are able to infect susceptibles. Such an outbreak may be expected to last as long as 1 month, with few fatalities in the susceptible population. Where this is not the case, a Class 2 outbreak occurs, and would be significantly harder for an external population to counter, due to the increased number of zombies present in the population. The duration of a Class 2 outbreak may be in the range of 15 days to 5 months. Such a Class 2 outbreak may then escalate to a Class 4 outbreak if the zombie population remains unchecked by some external force.

While a cure for the zombie virus as investigated in section 4.3.1 may provide a state in which humans and zombies may coexist, this is not a desirable outcome. Although, a cure for the zombie virus appears to be the only way to guarantee that human life continues in the event of a Class 4

outbreak, it is based on an unreasonable assumption, since such a cure would also be required to be able to resurrect the dead.

Throughout section 5, it is reiterated that the susceptible population must be able to destroy the zombies at a sufficient rate in order to survive. Whilst the introduction of a fighter or blaze class can appear to be beneficial to the susceptible population, neither class is able to guarantee the survival of the susceptible population. It is also suggested that the presence of an enhanced variation of zombie hinders the susceptible population's ability to overcome the zombie horde. However, section 5.4 finds that culling the susceptible population may be beneficial to the survival of an external population. Whilst some may consider that the needs of the many, outweigh the needs of the few, such a tactic would surely only be repeated until the population has destroyed itself through fear.

The simulations of the models suggest that the best way to prevent the possibility of a Class 4 outbreak occurring, is to counter the outbreak in the early stages of Class 1, hence never allowing a Class 2 outbreak to overwhelm the susceptible population. The zombie horde must be dealt with swiftly and effectively, whilst it still only consists of a small number of individuals, else the outbreak may escalate to a higher classification and bring about the extinction of the human race.

7. References

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8. Appendix

> Appendix 1

First, various commands must be used to access the tool kits.

$with(plots):$

$with(DEtools):$

$with(ColorTools):$

$with(VectorCalculus):$

$with(LinearAlgebra):$

The differential equations for the SI epidemic are then set up where des1 denotes the differential equation for S, and def1 denotes the differential equation for the infective population.

$des1 := diff(S(t), t) = -\beta \cdot S(t) \cdot F(t):$

$def1 := diff(F(t), t) = \beta \cdot S(t) \cdot F(t):$

Using the solve command returns the steady states for the model.

$solve([-\beta \cdot S(t) \cdot F(t) = 0, \beta \cdot S(t) \cdot F(t) = 0], [S(t), F(t)]):$

sic1 and fic1 denote the initial conditions based on the population for the respective classes. Whilst
sv1 and fv1 are where the values for parameters are substituted into the differential equations des1
and def1.

$sic1 := S(0) = 500:$

$fic1 := F(0) = 1:$

$sv1 := subs(\beta = 0.005, des1):$

$fv1 := subs(\beta = 0.005, def1):$

Using the dsolve command generates a numeric simulation for the model. This can then be displayed
graphically using the odeplot command.

$s1 := dsolve([sv1, sic1, fv1, fic1], [S(t), F(t)], numeric):$

$odeplot(s1, [[t, S(t)], [t, F(t)]], t = 0..10, color = [blue, red], legend = [S, I], labels = [\tau,$
$\text{"S, I"}]):$

The Jacobian matrix and its eigenvalues may be calculated using the commands below.

$j1 := Jacobian([-\beta \cdot S \cdot F, \beta \cdot S \cdot F], [S, F]):$

$\xrightarrow{\text{eigenvalues}}$

$j1s1 := subs([\beta = 0.005, S = S, F = 0], j1):$

$\xrightarrow{\text{eigenvalues}}$

$j1s2 := subs([\beta = 0.005, S = 0, F = F], j1):$

$\xrightarrow{\text{eigenvalues}}$

The phase plane plots can obtained by using the dsolve command with the relevant conditions, and then
using the odeplot command in the manner shown below. One iteration will generate one line for the phase
plane, whilst the initial conditions uic1 and vic1 may be altered to generate a full phase plane plot.

$deu1 := diff(u(t), t) = -r \cdot u(t) \cdot v(t):$

$dev1 := diff(v(t), t) = r \cdot u(t) \cdot v(t):$

$uv1 := subs(r = 1.5, deu1):$

$vv1 := subs(r = 1.5, dev1):$

$uic1 := u(0) = 0.9:$

$vic1 := v(0) = 0.1:$

$sol1 := dsolve([uv1, uic1, vv1, vic1], [u(t), v(t)], numeric) :$

> $plot1 := odeplot(sol1, [u(t), v(t)], t = 0..10) :$